IMA Journal of Applied Mathematics (2014) **79**, 502–534
doi:10.1093/imamat/hxt001
Advance Access publication on 4 February 2013

Explicit construction of compactly supported biorthogonal multiwavelets via matrix extension

Yi-Gang Cen and Rui-Zhen Zhao
Institute of Information Science, Beijing Jiaotong University, Beijing 100044, P. R. China and Beijing Key Laboratory of Advanced Information Science and Network Technology, Beijing 100044, P. R. China
cenyigang@163.com rzhzhao@bjtu.edu.cn

Li-Hui Cen*
School of Information Science and Engineering, Central South University, Hunan, Changsha 410083, P. R. China and Key Laboratory of System Control and Information Processing, Ministry of Education, Shanghai 200240, P. R. China
*Corresponding author: lhcen@csu.edu.cn

Zhen-Jiang Miao
Institute of Information Science, Beijing Jiaotong University, Beijing 100044, P. R. China and Beijing Key Laboratory of Advanced Information Science and Network Technology, Beijing 100044, P. R. China
zjmiao@bjtu.edu.cn

and

Xiao-Fang Chen
School of Information Science and Engineering, Central South University, Hunan, Changsha 410083, P. R. China
xiaofangchen@csu.edu.cn

[Received on 3 April 2011; revised on 29 November 2012; accepted on 9 January 2013]

Polyphase matrix extension of the scaling vector functions plays an important role in the construction of compactly supported biorthogonal multiwavelets. However, the involved computations are very complicated, and there is no unified, direct formula available so far. In this paper, abstract algebraic methods are used to investigate the canonical forms of polyphase matrices of the scaling vector functions. According to the related properties of canonical forms of polyphase matrices, it is proved that the matrix extension equations are always solvable so that two explicit formula groups of the solution set corresponding to different canonical forms can be derived. All the explicit formulas are represented via the submatrices of polyphase matrices directly. Furthermore, for a given matrix extension problem, any solution can be obtained from these explicit formulas via product-preserving transformations, which means that, the proposed algorithm provides a complete solution set. Computational examples demonstrated that by using the explicit formulas, our matrix extension algorithm is direct and effective. Finally, a simple application by using multiwavelets for denoising is presented and the experimental results showed that the multiwavelets outperformed the scalar wavelets under different test signals.

Keywords: compactly supported biorthogonal multiwavelets; abstract algebraic approach; canonical form; polyphase matrix extension; product-preserving transformation.

1. Introduction

Multiwavelets have been widely used in many applications such as denoising, image compression, prediction, watermarking and so on since they have a number of advantages over scalar wavelets. The main motivation of employing multiwavelets is that several desirable properties, such as orthogonality, symmetry and short support for a given vanishing moment, can be made full use of simultaneously in the applications. But these properties cannot be shared by the scalar wavelets except the Haar wavelet. However, the Haar wavelet suffers from a major disadvantage, the discontinuity in the spatial domain. Symmetry means that the filter bank can possess linear phase. In signal processing, a filter delays different frequency components of a signal by the same amount if the filter has linear phase (constant phase delay). In image processing, filters with non-linear phase can introduce artefacts that are visually annoying. The width of the supports of the filters is proportional to the number of high-amplitude wavelet coefficients created by a brutal transition, such as an edge. For a more accurate localization of singularities, the number of high-amplitude wavelet coefficients should be as small as possible. So the supports of the filters should be as short as possible. Moreover, the more the vanishing moments, the smaller are the coefficients that can be produced over smooth regions at fine scales. Therefore, the multiwavelet coefficients that belong to the noise components can be more easily distinguished at fine scales.

Multiwavelets have raised great interest among the research community and have been investigated intensively by scientists in the past couple of decades. The study of multiwavelets was first initiated by Goodman *et al.* (1990), and one of the earliest and most widely used multiwavelets is the so-called GHM multiwavelet (because it was constructed by Geronimo, Hardin and Massopust) see Geronimo *et al.* (1994); see also Donovan *et al.* (1996) by using the fractal interpolation method. By imposing the Hermite interpolation conditions, Chui & Lian (1996) constructed symmetric–antisymmetric orthonormal multiwavelets with a particular emphasis on the maximum number of vanishing moments. Then, a new method was developed to construct the orthogonal and biorthogonal multiwavelets via matrix extension (see Lawton *et al.*, 1996; Goh & Yap, 1998). By using paraunitary matrix extension, Lawton *et al.* (1996) proposed a construction method for orthogonal multiwavelets, and the basic idea was then extended to the construction of biorthogonal multiwavelets proposed by Goh & Yap (1998).

This paper focuses on the construction of biorthogonal multiwavelets. It is well known that there are explicit formulas for the construction of biorthogonal uniwavelets (i.e. scalar wavelets) (see Daubechies, 1992): if p_k and $\tilde{p}_k$ are the low-pass filters corresponding to a pair of biorthogonal uniscaling functions $\phi(x)$ and $\tilde{\phi}(x)$, respectively, then the associated high-pass filters q_k and $\tilde{q}_k$ corresponding to the uniwavelets $\psi(x)$ and $\tilde{\psi}(x)$, respectively, can be obtained via the following: $q_k = (-1)^{k-1}\tilde{p}_{1-k}$ and $\tilde{q}_k = (-1)^{k-1}p_{1-k}$. However, there are no such explicit relationships for the construction of biorthogonal multiwavelets. In this paper, an abstract algebraic approach for polyphase matrix extension is proposed to construct the compactly supported biorthogonal multiwavelets without any constraints. Specifically, the following properties hold.

(1) The canonical forms of polyphase matrices of the scaling vector functions are studied by using abstract algebraic method. It is proved that the polyphase matrices of the scaling vector functions for any compactly supported biorthogonal multiwavelets can be converted into canonical forms through finite steps of column–row product-preserving transformations.

(2) For the matrix extension problem, two solution sets with explicit formulas corresponding to different canonical forms are provided (Tables 1 and 2). They are represented by the submatrices

of the scaling polyphase matrices directly. Furthermore, since the extension result is not unique, we proved that any extension solution can be obtained from the explicit formulas given in Table 1 or 2 via finite steps of the product-preserving transformations.

(3) For some extension problems, if there are several expression sets of the solutions, appropriate selection of formulas can further decrease the computational cost to obtain the desired extension forms.

In order to further validate the proposed method, four examples were provided at the end of the paper and the experimental results demonstrated that our approach obtained the same results as those provided in Goh & Yap (1998), Strela & Walden (1998), Hardin & Marasovich (1999) and Tan *et al.* (1999). A denoising experiment was also given to test the constructed biorthogonal multiwavelets in this paper, and the result showed that multiwavelets generally outperformed scalar wavelets. The matrix extension problem of the compactly supported biorthogonal multiwavelets was first reviewed as follows.

Let $\Phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_p(x))^\top$ and $\tilde{\Phi}(x) = (\tilde{\phi}_1(x), \tilde{\phi}_2(x), \ldots, \tilde{\phi}_p(x))^\top$ be the two vector functions satisfying the matrix dilation equations $\Phi(x) = \sum_{k\in Z} H(k)\Phi(2x-k)$ and $\tilde{\Phi}(x) = \sum_{k\in Z} \tilde{H}(k)\tilde{\Phi}(2x-k)$, respectively. Here, $H(k)$ and $\tilde{H}(k)$ are finite two-scale matrix coefficients, which entries are real-valued numbers. We denote by Z the set composed of all integers. Suppose that subspaces $\{V_j\}_{j\in Z}$ and $\{\tilde{V}_j\}_{j\in Z}$ establish two multiresolution analyses with multiplicity p in $L^2(R)$, where $V_j = \overline{\text{span}}\{2^{j/2}\phi_i(2^j\cdot -k), 1 \leqslant i \leqslant p, k \in Z\}$ and $\tilde{V}_j = \overline{\text{span}}\{2^{j/2}\tilde{\phi}_i(2^j\cdot -k), 1 \leqslant i \leqslant p, k \in Z\}$. Note that $\Phi(x)$ and $\tilde{\Phi}(x)$ are a pair of compactly supported *biorthogonal scaling vector functions* if they satisfy the *biorthogonality condition*

$$\langle \Phi(\cdot), \tilde{\Phi}(\cdot - n)\rangle = \delta_{0,n} I_{p\times p},$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product, $I_{p\times p}$ is the identity matrix and $\delta_{i,j}$ is the Kronecker delta, i.e.

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Suppose that the vector-valued functions $\Psi = (\psi_1, \psi_2, \ldots, \psi_p)^\top$ and $\tilde{\Psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_p)^\top$ satisfy the dilation equations $\Psi(x) = \sum_{k\in Z} G(k)\Phi(2x-k)$ and $\tilde{\Psi}(x) = \sum_{k\in Z} \tilde{G}(k)\tilde{\Phi}(2x-k)$, where $G(k)$ and $\tilde{G}(k)$ are finite real-valued matrix coefficients. Define $W_j = \overline{\text{span}}\{2^{j/2}\psi_i(2^j\cdot -k), 1 \leqslant i \leqslant p, k \in Z\}$ and $\tilde{W}_j = \overline{\text{span}}\{2^{j/2}\tilde{\psi}_i(2^j\cdot -k), 1 \leqslant i \leqslant p, k \in Z\}$ such that W_j and $\tilde{W}_j, j \in Z$ are the complementary subspaces of V_j and $\tilde{V}_j$ in V_{j+1} and $\tilde{V}_{j+1}$, respectively. Then $\Psi(x)$ and $\tilde{\Psi}(x)$ are a pair of compactly supported *biorthogonal multiwavelets* associated with the scaling vector functions $\Phi(x)$ and $\tilde{\Phi}(x)$, if they satisfy the following *biorthogonality conditions*:

$$\begin{cases} \langle \Phi(\cdot), \tilde{\Psi}(\cdot - n)\rangle = \langle \Psi(\cdot), \tilde{\Phi}(\cdot - n)\rangle = 0, \\ \langle \Psi(\cdot), \tilde{\Psi}(\cdot - n)\rangle = \delta_{0,n} I_{p\times p}. \end{cases}$$

Define two $p \times p$ matrices $H_e(z)$ and $H_o(z)$ as

$$H_e(z) = \frac{1}{\sqrt{2}} \sum_{k\in Z} H(2k)z^k \quad \text{and} \quad H_o(z) = \frac{1}{\sqrt{2}} \sum_{k\in Z} H(2k+1)z^k,$$

where $z \in \{z : |z| = 1\}$ (the unit circle in the complex plane). The *polyphase matrix* of $\Phi(x)$ is then defined as $H(z) = (H_e(z)\ H_o(z))_{p\times 2p}$. Likewise, $\tilde{H}(z) = (\tilde{H}_e(z)\ \tilde{H}_o(z))_{p\times 2p}$, $G(z) = (G_e(z)\ G_o(z))_{p\times 2p}$ and $\tilde{G}(z) = (\tilde{G}_e(z)\ \tilde{G}_o(z))_{p\times 2p}$ are the polyphase matrices of $\tilde{\Phi}(x)$, $\Psi(x)$ and $\tilde{\Psi}(x)$, respectively. Let $R[z]$ be a set consisting of all real-coefficient Laurent polynomials in $z \in T$. Under the ordinary addition and multiplication of Laurent polynomials, $R[z]$ is a ring. It can be verified that the polyphase matrices $H(z)$, $\tilde{H}(z)$, $G(z)$ and $\tilde{G}(z)$ are all matrices over $R[z]$.

From the biorthogonality conditions of $\Phi(x)$, $\tilde{\Phi}(x)$, $\Psi(x)$ and $\tilde{\Psi}(x)$, we have

$$\begin{cases} H_e(z)\tilde{H}_e(z)^* + H_o(z)\tilde{H}_o(z)^* = I_{p\times p}, \\ H_e(z)\tilde{G}_e(z)^* + H_o(z)\tilde{G}_o(z)^* = \mathbb{O}_{p\times p}, \\ G_e(z)\tilde{H}_e(z)^* + G_o(z)\tilde{H}_o(z)^* = \mathbb{O}_{p\times p}, \\ G_e(z)\tilde{G}_e(z)^* + G_o(z)\tilde{G}_o(z)^* = I_{p\times p}, \end{cases}$$

where $\mathbb{O}_{p\times p}$ denotes the $p \times p$ zero matrix and the superscript $*$ denotes the complex-conjugate transpose. Equivalently,

$$Q(z)\tilde{Q}(z)^* = \begin{pmatrix} H_e(z) & H_o(z) \\ G_e(z) & G_o(z) \end{pmatrix}_{2p\times 2p} \begin{pmatrix} \tilde{H}_e(z)^* & \tilde{G}_e(z)^* \\ \tilde{H}_o(z)^* & \tilde{G}_o(z)^* \end{pmatrix}_{2p\times 2p} = I_{2p\times 2p}.$$

In this paper, we concentrate on the following matrix extension problem. For a given pair of compactly supported biorthogonal scaling vector functions $\Phi(x)$ and $\tilde{\Phi}(x)$ with polyphase matrices satisfying $H(z)\tilde{H}(z)^* = I_{p\times p}$, $H(z)$ and $\tilde{H}(z)$ are needed to be extended to two $2p \times 2p$ unimodular matrices (invertible matrices) $Q(z)$ and $\tilde{Q}(z)$ over $R[z]$ satisfying $Q(z)\tilde{Q}(z)^* = I_{2p\times 2p}$, such that the first p rows of $Q(z)$ and those of $\tilde{Q}(z)$ are the matrices $H(z)$ and $\tilde{H}(z)$, respectively. One way to tackle this problem is to solve $G_e(z)$, $G_o(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ from the matrix equation $Q(z)\tilde{Q}(z)^* = I_{2p\times 2p}$, which is investigated in this paper.

The rest of this paper is organized as follows. Section 2 provides a brief introduction of abstract algebra including the unimodular matrix over $R[z]$, elementary transformations, product-preserving transformations and the normal form of an $m \times n$ matrix over a Euclidean ring. In Section 3, we prove that, for any polyphase matrices $H(z)_{p\times 2p}$ and $\tilde{H}(z)^*_{2p\times p}$, $\operatorname{rank} H(z) = \operatorname{rank} \tilde{H}(z)^* = p$, they can be transformed into a canonical form by finite steps of product-preserving transformations. This is a fundamental result for the proposed matrix extension algorithm. Section 4 gives explicit formulas for computing the polyphase matrices $G(z)$ and $\tilde{G}(z)^*$ via the corresponding polyphase matrices $H(z)$ and $\tilde{H}(z)^*$. The general algorithm for constructing the compactly supported biorthogonal multiwavelets is then proposed. In Section 5, three examples are given to demonstrate that the proposed method can obtain the same results as those presented in Goh & Yap (1998), Strela & Walden (1998) and Hardin & Marasovich (1999). Section 6 concludes this paper.

2. Unimodular matrices over $R[z]$ and product-preserving transformation

The main idea of the proposed matrix extension approach is to solve the polyphase matrices $G_e(z)$, $G_o(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ over $R[z]$ based on the identity $Q(z)\tilde{Q}(z)^* = I_{2p\times 2p}$. Thus, it is inevitable to deal with the inverse of a matrix over $R[z]$, i.e. we need to investigate the *unimodular matrix* (invertible matrix) over $R[z]$. On the other hand, an important tool used in the proposed approach is the so-called *product-preserving transformation*, which ensures that we can modify the extension resultant matrices to obtain the desired forms for a particular application. So, this section will mainly discuss these two problems and related backgrounds.

The Laurent polynomial ring $R[z]$ has the following basic properties (see Jacobson, 1974, pp. 141–143):

(1) $R[z]$ is a commutative ring with the identity 1.

(2) All the polynomials in z form a Euclidean subring of $R[z]$, denoted as $P[z]$, under the ordinary addition and multiplication operations on $R[z]$.

(3) An element $a(z) \in R[z]$ is called *invertible* if $1/a(z) \in R[z]$. Moreover, $a(z)$ is an invertible element of $R[z]$, if, and only if, $a(z) = cz^k$, where $c \in R$ and $c \neq 0, k \in Z, z \in T$. An invertible element of $R[z]$ is also called a *unit*.

Proof. From the definition of $R[z]$, Properties (1) and (2) can be verified. Here, only the proof of Property (3) is given as follows.

If $\forall\, a(z) \in R[z]$ and $a(z) \neq 0$, then $a(z)$ can be expressed as $\sum_{i=N_1}^{N_2} a_i z^i$, where $i = N_1, \ldots, N_2$ and N_1, $N_2 \in Z$. Thus, $1/a(z) = 1/(\sum_{i=N_1}^{N_2} a_i z^i)$. Obviously, $1/a(z)$ remains to be a Laurent polynomial, if and only if, $\sum_{i=N_1}^{N_2} a_i z^i$ is a non-zero monomial in z (or $\bar{z} = 1/z$). Otherwise, if $a(z) = \sum_{i=N_1}^{N_2} a_i z^i$ and $a(z) \neq cz^k$ (where $c \in R$ but $c \neq 0$, $k \in Z$, $z \in T$), then $1/a(z)$ is only an element in the quotient field of $R[z]$ (i.e. the field of fractions on $R[z]$). □

Because the entries in the polyphase matrices are Laurent polynomials according to the definition of polyphase matrices in Section 1, it is necessary to investigate such kind of matrices with Laurent polynomial entries.

DEFINITION 2.1 $A(z) = (a_{ij}(z))_{m\times n}$ is called a matrix over $R[z]$ if all its entries lie in $R[z]$, i.e. every entry $a_{ij}(z)$ of $A(z)$ is a Laurent polynomial in $z \in T$. The set composed of all such $m \times n$ matrices over $R[z]$ is denoted by $M_{mn}(R[z])$. We simply denote $M_{p1}(R[z]) = \{(a_1(z), \ldots, a_p(z))^\top \mid a_i(z) \in R[z],\ i = 1, \ldots, p\}$ is simply denoted by M_p (see Jacobson, 1974, pp. 153–202; Roman, 1997, pp. 107–119).

Based on the Definition 2.1, for $\alpha(z) = (a_1(z), \ldots, a_p(z))^\top$ and $\beta(z) = (b_1(z), \ldots, b_p(z))^\top \in M_p$, $r(z) \in R[z]$, the *addition* and *scalar multiplication* in M_p are defined as

$$\alpha(z) + \beta(z) = (a_1(z) + b_1(z), \ldots, a_p(z) + b_p(z))^\top$$

and

$$r(z) \cdot \alpha(z) = (r(z) \cdot a_1(z), \ldots, r(z) \cdot a_p(z))^\top.$$

Then M_p is a $R[z]$-*module* (or a *module over* $R[z]$) under the above-mentioned addition and scalar multiplication.

The following Definition 2.2 to Theorem 2.1 presents the definitions and the necessary and sufficient conditions for a matrix over $R[z]$ to be invertible.

DEFINITION 2.2 If $A(z) = (a_{ij}(z))_{n\times n}$ is a square matrix over $R[z]$ and there exists a square matrix $B(z) = (b_{ij}(z))_{n\times n}$ over $R[z]$ such that $A(z)B(z) = B(z)A(z) = I_{n\times n}$, then $A(z)$ is called an *invertible matrix over* $R[z]$ and its inverse is denoted by $A^{-1}(z) = B(z)$. Likewise, $B^{-1}(z) = A(z)$. Moreover, if $A(z)$ is invertible over $R[z]$, then its inverse matrix is unique. An invertible matrix over $R[z]$ is also often referred to as a *unimodular square matrix* over $R[z]$ (see Waerden, 1978, p. 566).

LEMMA 2.1 If σ is a commutative ring with identity, a matrix over σ is invertible, if and only if, its determinant is invertible in σ.

The reader is referred to Jacobson (1974, Theorem 2, p. 94) to get more details about the proof of this lemma. From Lemma 2.1 and Property (3) of $R[z]$, we have the following necessary and sufficient conditions for a square matrix $A(z)$ over $R[z]$ to be invertible.

THEOREM 2.1 A matrix $A(z) = (a_{ij}(z))_{n\times n}$ over $R[z]$ is invertible, if and only if, $\det A(z) = cz^k$, where $c \in R$ but $c \neq 0$, $k \in Z$, $z \in T$.

It can be verified that the set composed of all unimodular square matrices of size $n \times n$ each, denoted by $\mathrm{GL}_n(R[z])$, is a group, under the ordinary matrix multiplication operation.

In the lifting scheme, the polyphase matrix of any complementary finite impulse response matrix filter pair (H, G) can always be factorized through elementary row and column transformations (see Goh *et al.*, 2000), which also play important roles in our matrix extension algorithm. But the elementary row (column) transformations and the multiwavelet construction approach proposed in this paper are different from the lifting scheme. Here, the definitions of elementary transformations and some related conclusions are presented as follows.

DEFINITION 2.3 $\forall A(z) = (a_{ij}(z))_{m\times n} \in M_{mn}(R[z])$, the following three types of operations are called *elementary transformations* of $A(z)$:

Type I. Interchange the ith row (column) and the jth row (column) of $A(z)$, $i \neq j$.

Type II. Multiply a row (column) of $A(z)$ by a unit of $R[z]$.

Type III. $\forall b \in R[z]$, multiply the jth row (ith column) of $A(z)$ by b and add it to the ith row (jth column), $i \neq j$.

DEFINITION 2.4 A square matrix is called an *elementary matrix* over $R[z]$ if it can be obtained from the identity matrix $I_{n\times n}$ by performing a single elementary operation. The resultant three types of elementary matrices corresponding to the elementary transformations are (see Jacobson, 1974, pp. 176–177) as follows:

Type I′. $P_{ij} = I_{n\times n} - e_{ii} - e_{jj} + e_{ij} + e_{ji}$, where e_{ij} is an $n \times n$ matrix whose (i, j)th element is 1, and all other elements are 0.

Type II′. Let u be a unit of $R[z]$, $D_i(u) = I_{n\times n} + (u - 1)e_{ii}$.

Type III′. $\forall b \in R[z]$ and $i \neq j$, $T_{ij}(b) = I_{n\times n} + be_{ij}$.

From the above definitions, the following fundamental facts will be utilized for further development.

PROPOSITION 2.1 Every $n \times n$ elementary matrix over $R[z]$ is an element of group $\mathrm{GL}_n(R[z])$. Elementary matrices P_{ij}, $D_i(u)$ and $T_{ij}(b)$ are invertible with $P_{ij}^{-1} = P_{ij}$, $D_i^{-1}(u) = D_i(u^{-1})$ and $T_{ij}^{-1}(b) = T_{ij}(-b)$, respectively.

PROPOSITION 2.2 Suppose that $A(z) = (a_{ij}(z))_{m\times n} \in M_{mn}(R[z])$; then a *row*-elementary transformation of $A(z)$ amounts to *left* multiplication of $A(z)$ by an $m \times m$ elementary matrix; likewise, a *column*-elementary transformation of $A(z)$ amounts to *right* multiplication of $A(z)$ by an $n \times n$ elementary matrix.

PROPOSITION 2.3 Suppose that $A(z)$ is a matrix over $R[z]$ and $A_1(z)$ is obtained from $A(z)$ by an elementary transformation. Then,

(1) $A_1(z)$ remains as a matrix over $R[z]$.

(2) $A_1(z)$ remains as a unimodular square matrix if $A(z)$ is a unimodular square matrix.

(3) $A_1(z)$ remains as a non-unimodular and non-singular matrix if $A(z)$ is a non-unimodular and non-singular matrix.

LEMMA 2.2 If $A \in M_{mn}(D)$, where D is a Euclidean ring, then A can be transformed into a normal form (diagonal form) by finite steps of elementary transformations:

$$\mathrm{diag}(d_1, \ldots, d_r, 0, \ldots, 0) = \begin{pmatrix} d_1 & & & & & \\ & \ddots & & & \mathbb{O} & \\ & & d_r & & & \\ & & & 0 & & \\ & \mathbb{O} & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

where $d_i \neq 0$ and $d_i | d_j$ (means that d_j is divisible by d_i) if $i \leqslant j$.

We refer the reader to Jacobson (1974, Theorem 3.8, pp. 176–198) for the proof of this lemma. In fact, when D is a Euclidean ring, only the elementary transformations of Type I and Type III are needed; hence, matrix A can be converted into its normal form.

LEMMA 2.3 Suppose that $A(z) = (a_{ij}(z))_{n\times n} \in \mathrm{GL}_n(R[z])$; then $A(z)$ can be converted into a matrix $B(z) = (b_{ij}(z))_{n\times n}$ over a Euclidean subring $P[z]$ of $R[z]$ through finite steps of row-elementary transformations of Type II and $B(z) \in \mathrm{GL}_n(R[z])$.

The proof is provided in Appendix A.

From Lemma 2.3, a matrix $A(z)$ over $R[z]$ can be converted into a matrix $B(z)$ over $P[z]$. Hence, $A(z)$ can be transformed into the diagonal form by Lemma 2.2. Furthermore, if $A(z)$ is a unimodular matrix, we have following theorem.

THEOREM 2.2 Suppose that $A(z) = (a_{ij}(z))_{n\times n} \in \mathrm{GL}_n(R[z])$; then $A(z)$ can be converted into the identity matrix $I_{n\times n}$ by finite steps of *row*-elementary transformations.

The proof of this theorem is presented in Appendix B. Similarly, if $A(z) \in \mathrm{GL}_n(R[z])$, then $A(z)$ can be converted into $I_{n\times n}$ through finite steps of *column*-elementary transformations.

DEFINITION 2.5 Suppose that $A(z) = (a_{ij}(z))_{m\times n} \in M_{mn}(R[z])$, $B(z) = (b_{ij}(z))_{n\times m} \in M_{nm}(R[z])$ satisfying $A(z)B(z) = I_{m\times m}$; then any pair of the following elementary transformations is called a *row–column* (or *column–row*) *product-preserving transformation* of $A(z)$ and $B(z)$.

Type I″. Interchange the ith row (column) and the jth row (column) of $A(z)$. Simultaneously interchange the ith column (row) and the jth column (row) of $B(z)$.

Type II$''$. Multiply the ith row (column) of $A(z)$ by a unit cz^k of $R[z]$ (where $c \in R$ but $c \neq 0$, and $k \in Z$). Simultaneously multiply the ith column (row) of $B(z)$ by $1/cz^k$.

Type III$''$. For any $b(z) \in R[z]$, multiply the ith row (column) of $A(z)$ by $b(z)$ and add it to the jth row (column) of $A(z)$. Simultaneously, multiply the jth column (row) of $B(z)$ by $(-b(z))$ and add it to the ith column (row) of $B(z)$.

THEOREM 2.3 Suppose that $A_1(z)$ and $B_1(z)$ are the matrices obtained from $A(z)$ and $B(z)$ as defined in Definition 2.5, respectively, by performing any type of product-preserving transformations. Then $A_1(z)$ and $B_1(z)$ remain as two matrices over $R[z]$ and $A_1(z)B_1(z) = I_{m\times m}$.

The proof of this theorem is given in Appendix C.

From Propositions 2.2, 2.3 and Theorems 2.2 and 2.3, the following useful corollary can be drawn, which is a generalization of the above product-preserving transformations.

COROLLARY 2.1 Suppose that $A(z) = (a_{ij}(z))_{m\times n} \in M_{mn}(R[z])$, $B(z) = (b_{ij}(z))_{n\times m} \in M_{nm}(R[z])$ satisfying $A(z)\cdot B(z) = I_{m\times m}$; then $\forall\, P(z) \in \mathrm{GL}_m(R[z])$, $(P(z)A(z))\cdot(B(z)P^{-1}(z)) = I_{m\times m}$, which is equivalent to the fact that $A(z)$ and $B(z)$ are performed by finite steps of row–column product-preserving transformations. Similarly, $\forall\, Q(z) \in \mathrm{GL}_n(R[z])$, $(A(z)Q(z))\cdot(Q^{-1}(z)B(z)) = I_{m\times m}$, which is equivalent to the fact that $A(z)$ and $B(z)$ are performed by finite steps of column–row product-preserving transformations.

This corollary will be used in Corollary 4.1 and Table 2 of Section 4.

As mentioned at the end of Section 1, in order to solve the polyphase matrices $G_e(z)$, $G_o(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ over $R[z]$ based on the identity $Q(z)\tilde{Q}(z)^* = I_{2p\times 2p}$, we need to study whether there are invertible matrices among $H_e(z)$, $H_o(z)$, $\tilde{H}_e(z)^*$ and $\tilde{H}_o(z)^*$. In fact, in the following section, we will prove that at least one among these four matrices is invertible (the polyphase matrices $H(z) = (H_e(z)\ H_o(z))$ and $\tilde{H}(z) = (\tilde{H}_e(z)\ \tilde{H}_o(z))$ with at least one invertible submatrix in them are called *canonical form*) by using the product-preserving transformations. This result ensures that the proposed matrix extension approach (see Section 4) can be applied successfully.

3. Canonical form of the polyphase matrices of scaling vector functions

In this section, we will concentrate on two topics; first, for any polyphase matrices $H(z)$ and $\tilde{H}(z)^*$, $\operatorname{rank} H(z) = \operatorname{rank} \tilde{H}(z)^* = p$; second, $H(z)$ and $\tilde{H}(z)^*$ can always be transformed into a canonical form by finite steps of column–row product-preserving transformations; i.e. $H_e(z)$, $H_o(z)$, $\tilde{H}_e(z)^*$ and $\tilde{H}_o(z)^*$ are all non-zero matrices and at least one of them is invertible.

LEMMA 3.1 $\alpha_1, \alpha_2, \ldots, \alpha_s \in M_p$ are $R[z]$-linearly independent, if and only if, their complex-conjugate transpose $\alpha_1^*, \alpha_2^*, \ldots, \alpha_s^*$ are $R[z]$-linearly independent.

Similarly, if $\alpha_1^*, \alpha_2^*, \ldots, \alpha_s^*$ are $R[z]$-linearly independent, then $\alpha_1, \alpha_2, \ldots, \alpha_s$ are also $R[z]$-linearly independent.

THEOREM 3.1 Suppose that

$$H(z) = (H_e(z)\ H_o(z))_{p\times 2p} \quad \text{and} \quad \tilde{H}(z) = (\tilde{H}_e(z)\ \tilde{H}_o(z))_{p\times 2p}$$

are the polyphase matrices of the scaling vector functions of certain compactly supported biorthogonal multiwavelets. Then the p row-vectors of $H(z)$ and $\tilde{H}(z)$ are $R[z]$-linearly independent, respectively.

Proof. Suppose that the p column-vectors of $\tilde{H}(z)^* = (\tilde{H}_e(z)\ \tilde{H}_o(z))^*_{2p\times p}$ are $\alpha_1^*, \ldots, \alpha_p^* \in M_{2p}$ and $\sum_{i=1}^{p} r_i\alpha_i^* = \mathbb{O}_{2p\times 1} \in M_{2p}$, where $r_i \in R[z]$, $i = 1, \ldots, p$. Thus, $H(z)\cdot(\sum_{i=1}^{p} r_i\alpha_i^*) = H(z)\cdot\mathbb{O}_{2p\times 1} = \mathbb{O}_{p\times 1}$. Moreover, $H(z)\cdot\tilde{H}(z)^* = I_{p\times p}$, i.e. $H(z)\cdot\alpha_i^* = e_i$, where $i = 1, \ldots, p$ and e_i is a $p\times 1$ vector whose ith element is 1, and all other elements are 0. Therefore,

$$H(z)\cdot\left(\sum_{i=1}^{p} r_i\alpha_i^*\right) = \sum_{i=1}^{p} r_i H(z)\cdot\alpha_i^* = \sum_{i=1}^{p} r_i e_i = (r_1, \ldots, r_p)^\top = \mathbb{O}_{p\times 1} \in M_p.$$

This leads to $r_i = 0$ for $i = 1, \ldots, p$, thus, $\alpha_1^*, \ldots, \alpha_p^*$ are $R[z]$-linearly independent. From Lemma 3.1, $\alpha_1, \ldots, \alpha_p$ are also $R[z]$-linearly independent, i.e. the p row-vectors of $\tilde{H}(z)$ are $R[z]$-linearly independent.

Similarly, from $\tilde{H}(z)H(z)^* = I_{p\times p}$, the p row-vectors of $H(z)$ are also $R[z]$-linearly independent. □

From the above discussions, we arrive at the following two results.

(1) The ranks of $H(z)$ and $\tilde{H}(z)^*$ are unchanged by elementary transformations.

(2) If $H(z)\tilde{H}(z)^* = I_{p\times p}$, then $\operatorname{rank} H(z) = \operatorname{rank}\tilde{H}(z)^* = p$.

In order to continue the discussion of the properties of polyphase matrices, the definition of canonical form is given as follows.

DEFINITION 3.1 Suppose that

$$A(z) = (A_1(z)\ A_2(z))_{p\times 2p} \quad \text{and} \quad \tilde{A}(z) = (\tilde{A}_1(z)\ \tilde{A}_2(z))_{p\times 2p}$$

are two matrices satisfying $A(z)\tilde{A}(z)^* = I_{p\times p}$ over $R[z]$. The matrix pair $A(z)$ and $\tilde{A}(z)^*$ are called *canonical* if $A_1(z), A_2(z), \tilde{A}_1(z)$ and $\tilde{A}_2(z)$ are all non-zero matrices and at least one among them is a unimodular square matrix, where $A_1(z)$, $A_2(z)$, $\tilde{A}_1(z)$ and $\tilde{A}_2(z)$ are all $p\times p$ matrices over $R[z]$.

THEOREM 3.2 Suppose that $H(z)$ and $\tilde{H}(z)$ are the polyphase matrices of the scaling vector functions of certain compactly supported biorthogonal multiwavelet; then $H(z)$ and $\tilde{H}(z)^*$ can be transformed into canonical forms by finite steps of column–row product-preserving transformations.

Proof. Since $\operatorname{rank} H(z) = \operatorname{rank}\tilde{H}(z)^* = p$, from Lemma 2.2, there are elementary matrices $P_1, \ldots, P_r$ (with $p\times p$ each) and $Q_1, \ldots, Q_s$ (with $2p\times 2p$ each) such that

$$P_r\cdots P_1 H(z) Q_1\cdots Q_s = \begin{pmatrix} d_1 & & 0 & \\ & \ddots & & \mathbb{O}_{p\times p} \\ 0 & & d_p & \end{pmatrix}_{p\times 2p},$$

denoted as $D(z) = (D_1(z)\ \mathbb{O}_{p\times p})_{p\times 2p}$, where $d_i \in R[z]$, $d_i \neq 0$ and $d_i | d_j$, if $i < j$.

Note that a pair of the elementary transformations corresponding to P_i and P_i^{-1} (for $i = 1, \ldots, r$) constitutes a row–column product-preserving transformation of $H(z)$ and $\tilde{H}(z)^*$. Similarly, a pair of the elementary transformations corresponding to Q_j and Q_j^{-1} (for $j = 1, \ldots, s$) constitutes a column–row product-preserving transformation of $H(z)$ and $\tilde{H}(z)^*$. Now, suppose that

$$Q_s^{-1}\cdots Q_1^{-1}\tilde{H}(z)^*P_1^{-1}\cdots P_r^{-1}=\tilde{D}(z)^*=\begin{pmatrix}\tilde{D}_1(z)^*\\ \tilde{D}_2(z)^*\end{pmatrix}_{2p\times p}.$$

Thus,

$$D(z)\cdot\tilde{D}(z)^*=I_{p\times p}\Rightarrow D_1(z)\cdot\tilde{D}_1(z)^*=\mathrm{diag}(d_1,\ldots,d_p)\cdot\tilde{D}_1(z)^*=I_{p\times p}.$$

From this identity, note that $D_1(z)$ and $\tilde{D}_1(z)^*$ are two $p\times p$ matrices over $R[z]$; therefore, $D_1(z)$ and $\tilde{D}_1(z)^*$ are two mutually inverse unimodular square matrices over $R[z]$. Thus,

$$\tilde{D}_1(z)^*=D_1(z)^{-1}=\mathrm{diag}(d_1,\ldots,d_p)^{-1}=\mathrm{diag}(1/d_1,\ldots,1/d_p).$$

Define $\tilde{d}_i^*=1/d_i$ for $i=1,\ldots,p$; then $\tilde{D}_1(z)^*=\mathrm{diag}(\tilde{d}_1^*,\ldots,\tilde{d}_p^*)$, where $\tilde{d}_i^*\in R[z]$ and $d_i\tilde{d}_i^*=1$. From Proposition 2.3, $\tilde{D}(z)^*$ remains as a matrix over $R[z]$, thus $\tilde{D}_2(z)^*$ is also a matrix over $R[z]$.

Define $P=P_r\cdots P_1$ and $Q=Q_1\cdots Q_s$; then $H(z)Q=(P^{-1}D_1(z)\ \ \mathbb{O})_{p\times 2p}$ and $Q^{-1}\tilde{H}(z)^*=(P^*\tilde{D}_1(z)\ \ P^*\tilde{D}_2(z))^*_{2p\times p}$, where $P^{-1}D_1(z)$ and $P^*\tilde{D}_1(z)$ are two $p\times p$ unimodular matrices over $R[z]$. Thus, $H(z)$ and $\tilde{H}(z)^*$ can be converted into $(P^{-1}D_1(z)\ \ \mathbb{O})_{p\times 2p}$ and $(P^*\tilde{D}_1(z)\ \ P^*\tilde{D}_2(z))^*_{2p\times p}$ by finite steps of column–row product-preserving transformations consisting of the matrix pairs $(Q_1,Q_1^{-1}),\ldots,(Q_s,Q_s^{-1})$. But $H(z)Q=(P^{-1}D_1(z)\ \ \mathbb{O})_{p\times 2p}$ and $Q^{-1}\tilde{H}(z)^*=(P^*\tilde{D}_1(z)\ \ P^*\tilde{D}_2(z))^*_{2p\times p}$ are *not* canonical, since it is required by Definition 3.1 that all the four submatrices of $H(z)Q$ and $Q^{-1}\tilde{H}(z)^*$ should be non-zero. Thus, we need to perform product-preserving transformations further until the definition of canonical form is satisfied. Two different cases are discussed as follows:

Case (1). If $\tilde{D}_2(z)\neq\mathbb{O}_{p\times p}$, the matrix pair $(P^{-1}D_1(z)\ \ \mathbb{O})_{p\times 2p}$ and $(P^*\tilde{D}_1(z)\ \ P^*\tilde{D}_2(z))^*_{2p\times p}$ can be converted into canonical form through a series of column–row product-preserving transformations as follows.

For any $b(z)\in R[z]$ and $b(z)\neq 0$, multiply the i_0th (for $1\leqslant i_0\leqslant p$) column of $(P^{-1}D_1(z)\ \ \mathbb{O})_{p\times 2p}$ by $b(z)$ and add it to its j_0th (for $p+1\leqslant j_0\leqslant 2p$) column and, simultaneously, multiply the j_0th row of $(P^*\tilde{D}_1(z)\ \ P^*\tilde{D}_2(z))^*_{2p\times p}$ by $(-b(z))$ and add it to its i_0th row. According to the above manipulations, the matrix pair $(P^{-1}D_1(z)\ \ \mathbb{O})_{p\times 2p}$ and $(P^*\tilde{D}_1(z)\ \ P^*\tilde{D}_2(z))^*_{2p\times p}$ are, in fact, converted into $(P^{-1}D_1(z)\ \ H_o'(z))_{p\times 2p}$ and $(\tilde{H}_e'(z)\ \ P^*\tilde{D}_2(z))^*_{2p\times p}$, respectively, where $H_o'(z)\neq\mathbb{O}$, $\tilde{H}_e'(z)\neq\mathbb{O}$ and $P^*\tilde{D}_2(z)\neq\mathbb{O}$. $P^{-1}D_1(z)$ is a unimodular square matrix and $(P^{-1}D_1(z)\ \ H_o'(z))_{p\times 2p}\cdot(\tilde{H}_e'(z)\ \ P^*\tilde{D}_2(z))^*_{2p\times p}=I_{p\times p}$. Hence, $(P^{-1}D_1(z)\ \ H_o'(z))_{p\times 2p}$ and $(\tilde{H}_e'(z)\ \ P^*\tilde{D}_2(z))^*_{2p\times p}$ are in canonical form.

Case (2). If $\tilde{D}_2(z)=\mathbb{O}_{p\times p}$, then the following column–row product-preserving transformations are performed on $(P^{-1}D_1(z)\ \ \mathbb{O})_{p\times 2p}$ and $(P^*\tilde{D}_1(z)\ \ P^*\tilde{D}_2(z))^*_{2p\times p}=(P^*\tilde{D}_1(z)\ \ \mathbb{O})^*_{2p\times p}$: $\forall\, a(z)\in R[z]$ and $b(z)\in R[z]$, $a(z)\neq 0$, $b(z)\neq 0$, taking $i_0\neq i_1$: $1\leqslant i_0\leqslant p$, $1\leqslant i_1\leqslant p$ and $j_0\neq j_1$: $p+1\leqslant j_0\leqslant 2p$, $p+1\leqslant j_1\leqslant 2p$.

Step 1. Multiply the i_0th column of $(P^{-1}D_1(z)\ \ \mathbb{O})_{p\times 2p}$ by $a(z)$ and add it to the j_0th column; simultaneously, multiply the j_0th row of $(P^*\tilde{D}_1(z)\ \ \mathbb{O})^*_{2p\times p}$ by $(-a(z))$ and add it to the i_0th row; then $(P^{-1}D_1(z)\ \ \mathbb{O})_{p\times 2p}$ and $(P^*\tilde{D}_1(z)\ \ \mathbb{O})^*_{2p\times p}$ are converted into $(P^{-1}D_1(z)\ \ H_o'(z))_{p\times 2p}$ and $(P^*\tilde{D}_1(z)\ \ \mathbb{O})^*_{2p\times p}$, respectively.

Step 2. Multiply the j_1th column of $(P^{-1}D_1(z)\ H'_o(z))_{p\times 2p}$ by $b(z)$ and add it to the i_1th column; simultaneously, multiply the i_1th row of $(P^*\tilde{D}_1(z)\ \mathbb{O})^*_{2p\times p}$ by $(-b(z))$ and add it to the j_1th row. As a result, the matrix pair $(P^{-1}D_1(z)\ H'_o(z))_{p\times 2p}$ and $(P^*\tilde{D}_1(z)\ \mathbb{O})^*_{2p\times p}$ are converted into $(P^{-1}D_1(z)\ H'_o(z))_{p\times 2p}$ and $(P^*\tilde{D}_1(z)\ \tilde{H}'_o(z))^*_{2p\times p}$, respectively.

After the above manipulations, the resultant four submatrices $P^{-1}D_1(z)$, $H'_o(z)$, $(P^*\tilde{D}_1(z))^*$ and $\tilde{H}'_o(z)^*$ are non-zero matrices and at least $P^{-1}D_1(z)$ is unimodular. Thus, $H(z)$ and $\tilde{H}(z)^*$ can be transformed into canonical form $(P^{-1}D_1(z)\ H'_o(z))_{p\times 2p}$ and $(P^*\tilde{D}_1(z)\ \tilde{H}'_o(z))^*_{2p\times p}$ by finite steps of column–row product-preserving transformations. □

In the proof of Theorem 3.2, it is only ensured that the first submatrix $H_e(z)$ in $H(z)$ is converted into a unimodular matrix $P^{-1}D_1(z)$. Moreover, by using the following equivalent relations:

$$
\begin{aligned}
H(z)\tilde{H}(z)^* &= (H_e(z)\ H_o(z))_{p\times 2p}\cdot(\tilde{H}_e(z)\ \tilde{H}_o(z))^*_{2p\times p} = I_{p\times p},\\
&\Leftrightarrow (\tilde{H}_e(z)\ \tilde{H}_o(z))_{p\times 2p}\cdot(H_e(z)\ H_o(z))^*_{2p\times p} = I_{p\times p},\\
&\Leftrightarrow (H_o(z)\ H_e(z))_{p\times 2p}\cdot(\tilde{H}_o(z)\ \tilde{H}_e(z))^*_{2p\times p} = I_{p\times p},\\
&\Leftrightarrow (\tilde{H}_o(z)\ \tilde{H}_e(z))_{p\times 2p}\cdot(H_o(z)\ H_e(z))^*_{2p\times p} = I_{p\times p},
\end{aligned}
$$

$H(z)$ and $\tilde{H}(z)^*$ can be converted into any canonical form by finite steps of column–row product-preserving transformations, so that any submatrix among $H_e(z)$, $H_o(z)$, $\tilde{H}_e(z)^*$ and $\tilde{H}_o(z)^*$ can be transformed into a unimodular matrix over $R[z]$.

In fact, in many cases, for any given $H(z)$ and $\tilde{H}(z)^*$, there is at least one unimodular matrix among the four submatrices $H_e(z)$ $H_o(z)$, $\tilde{H}_e(z)^*$ and $\tilde{H}_o(z)^*$. In the worst situation (i.e. no unimodular matrix among these four submatrices), $\operatorname{rank} H(z) = \operatorname{rank}\tilde{H}(z)^* = p$ according to Theorem 3.1. Thus, $H(z)$ and $\tilde{H}(z)^*$ can always be transformed into canonical forms with at least one unimodular submatrix according to Theorem 3.2. This property of polyphase matrices exactly ensures that the matrix extension problem is always solvable for any compactly supported biorthogonal multiwavelets. For the sake of simplicity, in the following discussion, we always assume that $H(z)$ and $\tilde{H}(z)^*$ are canonical.

4. Solution of the matrix extension problem

In this section, a new approach for conducting matrix extension of compactly supported biorthogonal multiwavelets is proposed. It provides explicit formulas in terms of the submatrices of $H(z)$ and $\tilde{H}(z)^*$ as follows.

THEOREM 4.1 Suppose that $H(z) = (H_e(z)\ H_o(z))_{p\times 2p}$ and $\tilde{H}(z)^* = (\tilde{H}_e(z)\ \tilde{H}_o(z))^*_{2p\times p}$ are the polyphase matrices of a certain compactly supported biorthogonal scaling vector functions. They are in a canonical form (here, $\tilde{H}_e(z)^*$ is assumed as a unimodular matrix) satisfying $H(z)\tilde{H}(z)^* = H_e(z)\tilde{H}_e(z)^* + H_o(z)\tilde{H}_o(z)^* = I_{p\times p}$. Then

$$
\begin{cases}
G_e(z) = -\tilde{H}_o(z)^*[\tilde{H}_e(z)^*]^{-1},\\
G_o(z) = I_{p\times p},\\
\tilde{G}_e(z)^* = -\tilde{H}_e(z)^* H_o(z),\\
\tilde{G}_o(z)^* = I_{p\times p} - \tilde{H}_o(z)^* H_o(z)
\end{cases}
$$

TABLE 1 *Solution sets obtained under different conditions (in which one unimodular matrix is imposed)*

Conditions	$G_e(z)$	$G_o(z)$	$\tilde{G}_e(z)^*$	$\tilde{G}_o(z)^*$
$\tilde{H}_o(z)^*$ is unimodular	$I_{p\times p}$	$-\tilde{H}_e(z)^*[\tilde{H}_o(z)^*]^{-1}$	$I-\tilde{H}_e(z)^*H_e(z)$	$-\tilde{H}_o(z)^*H_e(z)$
$\tilde{H}_e(z)^*$ is unimodular	$-\tilde{H}_o(z)^*[\tilde{H}_e(z)^*]^{-1}$	$I_{p\times p}$	$-\tilde{H}_e(z)^*H_o(z)$	$I-\tilde{H}_o(z)^*H_o(z)$
$H_o(z)$ is unimodular	$I-\tilde{H}_e(z)^*H_e(z)$	$-\tilde{H}_e(z)^*H_o(z)$	$I_{p\times p}$	$-[H_o(z)]^{-1}H_e(z)$
$H_e(z)$ is unimodular	$-\tilde{H}_o(z)^*H_e(z)$	$I-\tilde{H}_o(z)^*H_o(z)$	$-[H_e(z)]^{-1}H_o(z)$	$I_{p\times p}$

satisfies the following matrix equation:

$$
\begin{aligned}
Q(z)\tilde{Q}(z)^* &= \begin{pmatrix} H_e(z) & H_o(z) \\ -\tilde{H}_o(z)^*[\tilde{H}_e(z)^*]^{-1} & I_{p\times p} \end{pmatrix}_{2p\times 2p} \begin{pmatrix} \tilde{H}_e(z)^* & -\tilde{H}_e(z)^*H_o(z) \\ \tilde{H}_o(z)^* & I_{p\times p}-\tilde{H}_o(z)^*H_o(z) \end{pmatrix}_{2p\times 2p} \\
&= I_{2p\times 2p}.
\end{aligned}
$$

Proof. Because $\tilde{H}_e(z)^*$ is a unimodular square matrix over $R[z]$, $[\tilde{H}_e(z)^*]^{-1}$ remains as a unimodular square matrix over $R[z]$. Thus, $G_e(z)$, $G_o(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ are matrices over $R[z]$. Hence, $Q(z)$ and $\tilde{Q}(z)^*$ are two $2p\times 2p$ matrices over $R[z]$. Now, we need to verify that $Q(z)\tilde{Q}(z)^*=I_{2p\times 2p}$, by using the following four equations.

(1) Based on the imposed condition $H(z)\tilde{H}(z)^*=I_{p\times p}$, i.e. $H_e(z)\tilde{H}_e(z)^*+H_o(z)\tilde{H}_o(z)^*=I_{p\times p}$.

(2) $-H_e(z)\tilde{H}_e(z)^*H_o(z)+H_o(z)[I_{p\times p}-\tilde{H}_o(z)^*H_o(z)]$
$=-[I_{p\times p}-H_o(z)\tilde{H}_o(z)^*]H_o(z)+H_o(z)[I_{p\times p}-\tilde{H}_o(z)^*H_o(z)]$
$=-H_o(z)+H_o(z)\tilde{H}_o(z)^*H_o(z)+H_o(z)-H_o(z)\tilde{H}_o(z)^*H_o(z)$
$=O_{p\times p}$.

(3) $-\tilde{H}_o(z)^*[\tilde{H}_e(z)^*]^{-1}\tilde{H}_e(z)^*+\tilde{H}_o(z)^*=O_{p\times p}$.

(4) $\tilde{H}_o(z)^*[\tilde{H}_e(z)^*]^{-1}\tilde{H}_e(z)^*H_o(z)+[I_{p\times p}-\tilde{H}_o(z)^*H_o(z)]=I_{p\times p}$.

Therefore, $Q(z)\tilde{Q}(z)^*=I_{2p\times 2p}$.

Moreover, from Definition 2.2 in Section 2, $\tilde{Q}(z)^*Q(z)=I_{2p\times 2p}$, since $Q(z)$ and $\tilde{Q}(z)^*$ are two mutually inverse square matrices. □

In Theorem 4.1, only $\tilde{H}_e(z)^*$ is assumed to be a unimodular matrix. Furthermore, if $H_e(z)$, $H_o(z)$ or $\tilde{H}_o(z)^*$ is unimodular, which corresponds to other three canonical form types. Explicit formulas for the matrix extension problem can be obtained similar to that in Theorem 4.1, and they are listed in the Table 1.

In general, given $H(z)=(H_e(z)\ H_o(z))_{p\times 2p}$ and $\tilde{H}(z)^*=(\tilde{H}_e(z)\ \tilde{H}_o(z))^*_{2p\times p}$, the solution set of

$$
Q(z)\tilde{Q}(z)^* = \begin{pmatrix} H_e(z) & H_o(z) \\ G_e(z) & G_o(z) \end{pmatrix}_{2p\times 2p} \begin{pmatrix} \tilde{H}_e(z)^* & \tilde{G}_e(z)^* \\ \tilde{H}_o(z)^* & \tilde{G}_o(z)^* \end{pmatrix}_{2p\times 2p} = I_{2p\times 2p}
$$

is not unique. The following theorem presents the relationship between any two different extensions.

THEOREM 4.2 Suppose that $\{G_e(z), G_o(z), \tilde{G}_e(z)^*, \tilde{G}_o(z)^*\}$ and $\{G'_e(z), G'_o(z), \tilde{G}'_e(z)^*, \tilde{G}'_o(z)^*\}$ are two sets of matrices over $R[z]$ satisfying the matrix equations $Q(z)\tilde{Q}(z)^* = I_{2p\times 2p}$ and $Q'(z)\tilde{Q}'(z)^* = I_{2p\times 2p}$, respectively; then

$$Q'(z) = \begin{pmatrix} H_e(z) & H_o(z) \\ G'_e(z) & G'_o(z) \end{pmatrix}_{2p\times 2p} \quad \text{and} \quad \tilde{Q}'(z)^* = \begin{pmatrix} \tilde{H}_e(z)^* & \tilde{G}'_e(z)^* \\ \tilde{H}_o(z)^* & \tilde{G}'_o(z)^* \end{pmatrix}_{2p\times 2p}$$

can be obtained from

$$Q(z) = \begin{pmatrix} H_e(z) & H_o(z) \\ G_e(z) & G_o(z) \end{pmatrix}_{2p\times 2p} \quad \text{and} \quad \tilde{Q}(z)^* = \begin{pmatrix} \tilde{H}_e(z)^* & \tilde{G}_e(z)^* \\ \tilde{H}_o(z)^* & \tilde{G}_o(z)^* \end{pmatrix}_{2p\times 2p},$$

respectively, by finite steps of row–column product-preserving transformations of $Q(z)$ and $\tilde{Q}(z)^*$.

Proof. Since $Q(z)$ and $Q'(z)$ (similarly, $\tilde{Q}(z)^*$and $\tilde{Q}'(z)^*$) are $2p \times 2p$ unimodular square matrices, by Theorem 2.2 there are elementary matrices $q_1, \ldots, q_r$ and $q'_1, \ldots, q'_r$ each, with the size of $2p \times 2p$, such that

$$q_1 \cdots q_r Q(z) = I_{2p\times 2p} \quad \text{and} \quad q'_1 \cdots q'_s Q'(z) = I_{2p\times 2p}.$$

Hence,

$$Q'(z) = (q'_s)^{-1} \cdots (q'_1)^{-1} I_{2p\times 2p} = (q'_s)^{-1} \cdots (q'_1)^{-1} q_1 \cdots q_r Q(z).$$

Rewrite $(q'_s)^{-1}, \ldots, (q'_1)^{-1}, q_1, \ldots, q_r$ as $P_1, \ldots, P_k$ (where $k = r + s$); then $Q'(z) = P_1 \cdots P_k Q(z)$.

Consider the following transformation matrix pairs $(P_1, P_1^{-1}), (P_2, P_2^{-1}), \ldots, (P_k, P_k^{-1})$; then

$$Q(z)\tilde{Q}(z)^* = I_{2p\times 2p} \Rightarrow P_1 \cdots P_k Q(z)\tilde{Q}(z)^* P_k^{-1} \cdots P_1^{-1} = I_{2p\times 2p}.$$

Since $P_1 \cdots P_k Q(z) = Q'(z)$, according to the uniqueness of the inverse matrix of a unimodular square matrix and the condition $Q'(z)\tilde{Q}'(z)^* = I_{2p\times 2p}$, we should have $\tilde{Q}'(z)^* = \tilde{Q}(z)^* P_k^{-1} \cdots P_1^{-1}$. Thus,

$$P_1 \cdots P_k Q(z)\tilde{Q}(z)^* P_k^{-1} \cdots P_1^{-1} = Q'(z)\tilde{Q}'(z)^* = I_{2p\times 2p},$$

i.e. for any two different matrix extension forms $\{Q(z), \tilde{Q}(z)^*\}$ and $\{Q'(z), \tilde{Q}'(z)^*\}$ of $(H_e(z)\ H_o(z))_{p\times 2p}$ and $(\tilde{H}_e(z)\ \tilde{H}_o(z))^*_{2p\times p}$, respectively, they can be converted reciprocally by finite steps of row–column product-preserving transformations, and each of these product-preserving transformations only acts on a row of $(G_e(z)\ G_o(z))_{p\times 2p}$ and the corresponding column of $(\tilde{G}_e(z)\ \tilde{G}_o(z))^*_{2p\times p}$ without changing $(H_e(z)\ H_o(z))_{p\times 2p}$ and $(\tilde{H}_e(z)\ \tilde{H}_o(z))^*_{2p\times p}$. □

COROLLARY 4.1 Suppose that $H(z) = (H_e(z)\ \ H_o(z))_{p\times 2p}$ and $\tilde{H}(z)^* = (\tilde{H}_e(z)\ \ \tilde{H}_o(z))^*_{2p\times p}$ are the polyphase matrices of the scaling vector functions of certain compactly supported biorthogonal multiwavelets and $\tilde{H}_e(z)^*$ and $\tilde{H}_o(z)^*$ are unimodular square matrices; then

$$\begin{cases} G_e(z) = -[\tilde{H}_e(z)^*]^{-1}, \\ G_o(z) = [\tilde{H}_o(z)^*]^{-1}, \\ \tilde{G}_e(z)^* = -\tilde{H}_e(z)^* H_o(z) \tilde{H}_o(z)^*, \\ \tilde{G}_o(z)^* = (I_{p\times p} - \tilde{H}_o(z)^* H_o(z))\tilde{H}_o(z)^* = \tilde{H}_o(z)^* H_e(z) \tilde{H}_e(z)^* \end{cases}$$

satisfies the matrix equation

$$\begin{pmatrix} H_e(z) & H_o(z) \\ G_e(z) & G_o(z) \end{pmatrix}_{2p\times 2p} \begin{pmatrix} \tilde{H}_e(z)^* & \tilde{G}_e(z)^* \\ \tilde{H}_o(z)^* & \tilde{G}_o(z)^* \end{pmatrix}_{2p\times 2p} = I_{2p\times 2p}.$$

This corollary can be directly deduced from Corollary 2.1 and Theorems 4.1 and 4.2.

According to Corollary 4.1, the formulas in Table 1 lead to other forms under different conditions, which are listed in Table 2.

Tables 1 and 2 present two explicit formula sets of the solutions for the matrix extension problem, which give us more freedom to obtain different extension forms. When a pair of polyphase matrices $H(z) = (H_e(z)\ H_o(z))$ and $\tilde{H}(z)^* = (\tilde{H}_e(z)\ \tilde{H}_o(z))^*$ is given, we need to verify which one among $H_e(z)$, $H_o(z)$, $\tilde{H}_e(z)^*$ and $\tilde{H}_o(z)^*$ is unimodular (or can be converted into a unimodular matrix by the column–row product-preserving transformations). Then the computational formulas can be chosen from Table 1 or 2. If there are several (up to four) unimodular matrices among these four matrices, different sets of explicit formulas can be chosen from Tables 1 and 2. Thus, we can select one of them to obtain an extension result that is more closer to the desired extension form.

From Theorems 4.1, 4.2 and Tables 1 and 2, under the condition that $H(z)$ and $\tilde{H}(z)^*$ are of canonical form, the following algorithm is obtained for the matrix extension problem of compactly supported biorthogonal multiwavelets:

Step 1. Determine which submatrices in $H(z)$ and $\tilde{H}(z)^*$ are unimodular matrices; then the corresponding formulas can be selected from Table 1 or 2 to compute $G_e(z)$, $G_o(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$.

Step 2. In order to obtain the desired extension forms, appropriate row–column product-preserving transformations for $G(z) = (G_e(z)\ G_o(z))_{p\times 2p}$ and $\tilde{G}(z)^* = (\tilde{G}_e(z)\ \tilde{G}_o(z))^*_{2p\times p}$ are performed such that they can be converted into desired forms, respectively.

If the polyphase matrices are not in canonical form, i.e. no unimodular submatrices in $H(z)$ and $\tilde{H}(z)^*$, then they need to be converted into a canonical form. According to Theorem 3.2, we have the following algorithm:

Step 1. Transform $H(z)$ and $\tilde{H}(z)^*$ into a certain canonical form $H'(z) = (H'_e(z)\ H'_o(z))_{p\times 2p}$ and $\tilde{H}'(z)^* = (\tilde{H}'_e(z)\ \tilde{H}'_o(z))^*_{2p\times p}$ by finite steps of column–row product-preserving transformations (denoted as $\sigma_1, \sigma_2, \ldots, \sigma_s$).

Step 2. For the canonical form of $H'(z)$ and $\tilde{H}'(z)^*$ obtained in Step 1, the corresponding formulas can be selected from Table 1 or 2 to compute $G'_e(z)$, $G'_o(z)$, $\tilde{G}'_e(z)^*$ and $\tilde{G}'_o(z)^*$; then we have

$$Q'(z)\tilde{Q}'(z)^* = \begin{pmatrix} H'_e(z) & H'_o(z) \\ G'_e(z) & G'_o(z) \end{pmatrix}_{2p\times 2p} \cdot \begin{pmatrix} \tilde{H}'_e(z)^* & \tilde{G}'_e(z)^* \\ \tilde{H}'_o(z)^* & \tilde{G}'_o(z)^* \end{pmatrix}_{2p\times 2p} = I_{2p\times 2p}.$$

Step 3. Perform the column–row product-preserving transformations: $\sigma_s^{-1}, \sigma_{s-1}^{-1}, \ldots, \sigma_1^{-1}$ on the matrix pair $Q'(z)$ and $\tilde{Q}'(z)^*$; then the extension matrices $Q(z)$ and $\tilde{Q}(z)^*$ of $H(z)$ and $\tilde{H}(z)^*$ can be obtained, respectively, i.e.

$$Q(z)\tilde{Q}(z)^* = \begin{pmatrix} H_e(z) & H_o(z) \\ G_e(z) & G_o(z) \end{pmatrix}_{2p\times 2p} \cdot \begin{pmatrix} \tilde{H}_e(z)^* & \tilde{G}_e(z)^* \\ \tilde{H}_o(z)^* & \tilde{G}_o(z)^* \end{pmatrix}_{2p\times 2p} = I_{2p\times 2p}.$$

TABLE 2 *Solution sets obtained under different conditions (in which two unimodular matrices are imposed)*

Conditions	$G_e(z)$	$G_o(z)$	$\tilde{G}_e(z)^*$	$\tilde{G}_o(z)^*$
$\tilde{H}_o(z)^*$ and $\tilde{H}_e(z)^*$ are unimodular	$-[\tilde{H}_e(z)^*]^{-1}$	$[\tilde{H}_o(z)^*]^{-1}$	$-\tilde{H}_e(z)^* H_o(z)\tilde{H}_o(z)^*$	$\tilde{H}_o(z)^* H_e(z)\tilde{H}_e(z)^*$
$\tilde{H}_o(z)^*$ and $H_e(z)$ are unimodular	$H_e(z)$	$-H_e(z)\tilde{H}_e(z)^*[\tilde{H}_o(z)^*]^{-1}$	$H_e(z)^{-1} - \tilde{H}_e(z)^*$	$-\tilde{H}_o(z)^*$
$\tilde{H}_e(z)^*$ and $H_o(z)$ are unimodular	$-[H_o(z)]^{-1}\tilde{H}_o(z)^*[\tilde{H}_e(z)^*]^{-1}$	$[H_o(z)]^{-1}$	$-\tilde{H}_e(z)^*$	$[H_o(z)]^{-1} - \tilde{H}_o(z)^*$
$H_o(z)$ and $H_e(z)$ are unimodular	$H_o(z)\tilde{H}_o(z)^* H_e(z)$	$-H_e(z)\tilde{H}_e(z)^* H_o(z)$	$[H_e(z)]^{-1}$	$-[H_o(z)]^{-1}$
$H_o(z)$ and $\tilde{H}_e(z)^*$ are unimodular	$[\tilde{H}_e(z)^*]^{-1} - H_e(z)$	$-H_o(z)$	$\tilde{H}_e(z)^*$	$-[H_o(z)]^{-1} H_e(z)\tilde{H}_e(z)^*$
$H_e(z)$ and $\tilde{H}_o(z)^*$ are unimodular	$-H_e(z)$	$[\tilde{H}_o(z)^*]^{-1} - H_o(z)$	$-[H_o(z)]^{-1} H_o(z)\tilde{H}_o(z)^*$	$\tilde{H}_o(z)^*$

Step 4. In order to obtain the desired extension forms, appropriate row–column product-preserving transformations are performed for $G(z) = (G_e(z)\ G_o(z))_{p\times 2p}$ and $\tilde{G}(z)^* = (\tilde{G}_e(z)\ \tilde{G}_o(z))^*_{2p\times p}$ such that they can be converted into the desired forms, respectively.

5. Examples

Usually, for any given polyphase matrices $H(z)$ and $\tilde{H}(z)^*$ with canonical form, if the number of unimodular square matrices among the submatrices $H_e(z)$, $H_o(z)$, $\tilde{H}_e(z)^*$ and $\tilde{H}_o(z)^*$ is greater than or equal to 2, different formulas can be found in Tables 1 and 2 to compute $G_e(z)$, $G_o(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$. However, computational examples show that appropriate selection of formulas from these two tables can have explicit advantages. In Example 2, a comparison is given by using two sets of formulas selected from Tables 1 and 2, respectively, for the same extension problem, which demonstrates that appropriately selecting formulas can further decrease the computational cost.

Example 1 Here, the biorthogonal multiwavelet Bighm is reconstructed from its corresponding scaling vector functions, which are constructed in Strela & Walden (1998) by using a two-scale similarity transform. The scaling functions $\phi_1(x)$ and $\phi_2(x)$ have approximation order 1, and the dual ones $\tilde{\phi}_1(x)$, $\tilde{\phi}_2(x)$ have approximation order 3. The scaling matrix coefficients are given as follows:

$$H_0 = \begin{pmatrix} -\frac{1}{20} & \frac{1}{20} \\ -\frac{1}{20} & \frac{1}{20} \end{pmatrix}, \quad H_1 = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & -1 \end{pmatrix}, \quad H_2 = \begin{pmatrix} \frac{11}{10} & 0 \\ 0 & \frac{11}{10} \end{pmatrix}, \quad H_3 = \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & -1 \end{pmatrix},$$

$$H_4 = \begin{pmatrix} -\frac{1}{20} & -\frac{1}{20} \\ \frac{1}{20} & \frac{1}{20} \end{pmatrix}, \quad \tilde{H}_0 = \frac{1}{20}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{H}_1 = \frac{1}{20}\begin{pmatrix} 10 & -4 \\ 15 & -7 \end{pmatrix},$$

$$\tilde{H}_2 = \frac{1}{20}\begin{pmatrix} 20 & 0 \\ 0 & 10 \end{pmatrix}, \quad \tilde{H}_3 = \frac{1}{20}\begin{pmatrix} 10 & 4 \\ -15 & -7 \end{pmatrix}.$$

Solution. From the definition of polyphase matrix described in Section 1, we have

$$H_e(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{1}{20} + \frac{11}{10}z - \frac{1}{20}z^2 & \frac{1}{20} - \frac{1}{20}z^2 \\ -\frac{1}{20} + \frac{1}{20}z^2 & \frac{1}{20} + \frac{11}{10}z + \frac{1}{20}z^2 \end{pmatrix}, \quad H_o(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{2} + \frac{1}{2}z & -1 + z \\ \frac{1}{2} - \frac{1}{2}z & -1 - z \end{pmatrix},$$

$$\tilde{H}_e(z)^* = \frac{1}{20\sqrt{2}}\begin{pmatrix} 20z^{-1} & 0 \\ 0 & 10z^{-1} \end{pmatrix}, \quad \tilde{H}_o(z)^* = \frac{1}{20\sqrt{2}}\begin{pmatrix} 10(1+z^{-1}) & 15(1-z^{-1}) \\ 4(z^{-1}-1) & -7(1+z^{-1}) \end{pmatrix}.$$

Step 1. Since $\det \tilde{H}_e(z)^* = \frac{1}{4}z^{-2}$, $\tilde{H}_e(z)^*$ is a unimodular square matrix over $R[z]$ according to the Theorem 2.1, so $G_e(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ can be calculated by using the formulas in the second row of Table 1 directly.

Step 2 Compute $G_e(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ as follows:

$$G_e(z) = -\tilde{H}_o(z)^*[\tilde{H}_e(z)^*]^{-1} = -\frac{1}{20\sqrt{2}}\begin{pmatrix} 10(1+z^{-1}) & 15(1-z^{-1}) \\ 4(z^{-1}-1) & -7(1+z^{-1}) \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 2z & 0 \\ 0 & 4z \end{pmatrix}$$
$$= -\frac{1}{10}\begin{pmatrix} 5(z+1) & 15(z-1) \\ 2(1-z) & -7(z+1) \end{pmatrix},$$
$$\tilde{G}_e(z)^* = -\tilde{H}_e(z)^* H_o(z) = -\frac{1}{20\sqrt{2}}\begin{pmatrix} 20z^{-1} & 0 \\ 0 & 10z^{-1} \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{2}(1+z) & z-1 \\ \frac{1}{2}(1-z) & -1-z \end{pmatrix}$$
$$= -\frac{1}{4}\begin{pmatrix} 1+z^{-1} & 2(1-z^{-1}) \\ \frac{1}{2}(z^{-1}-1) & -1-z^{-1} \end{pmatrix},$$
$$\tilde{G}_o(z)^* = I_{2\times 2} - \tilde{H}_o(z)^* H_o(z) = I_{2\times 2} - \frac{1}{20\sqrt{2}}\begin{pmatrix} 10(1+z^{-1}) & 15(1-z^{-1}) \\ 4(z^{-1}-1) & -7(1+z^{-1}) \end{pmatrix}$$
$$\cdot \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{2}(1+z) & z-1 \\ \frac{1}{2}(1-z) & -1-z \end{pmatrix}$$
$$= \frac{1}{80}\begin{pmatrix} 30+5z+5z^{-1} & 10(z-z^{-1}) \\ 3(z^{-1}-z) & 36-6z-6z^{-1} \end{pmatrix}.$$

Step 3. Perform row–column product-preserving transformations for $(G_e(z)\ I_{2\times 2})_{2\times 4}$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))^*_{4\times 2}$ in order to obtain the same multiwavelet matrix coefficients as that in Strela & Walden (1998).

In order to describe the exact row–column product-preserving transformations involved, the following short-hand notation is used: the ith row (column) of a matrix is denoted by r_i (c_i), $i = 1, 2$, while (I) and (II) denote the first and second matrices, respectively.

$$(G_e(z)\ I_{2\times 2})_{2\times 4}\begin{pmatrix} \tilde{G}_e(z)^* \\ \tilde{G}_o(z)^* \end{pmatrix}_{4\times 2} = \begin{pmatrix} \frac{-(z+1)}{2} & \frac{3(1-z)}{2} & 1 & 0 \\ \frac{(z-1)}{5} & \frac{7(z+1)}{10} & 0 & 1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} \frac{-(1+z^{-1})}{4} & \frac{(z^{-1}-1)}{2} \\ \frac{(1-z^{-1})}{8} & \frac{(1+z^{-1})}{4} \\ \frac{(30+5z+5z^{-1})}{80} & \frac{(z-z^{-1})}{8} \\ \frac{3(z^{-1}-z)}{80} & \frac{(36-6z-6z^{-1})}{80} \end{pmatrix}$$

$$\xrightarrow[(\mathrm{II}):c_1\times\frac{1}{\sqrt{2}}z^{-1},c_2\times\frac{1}{\sqrt{2}}z^{-1}]{(\mathrm{I}):r_1\times\sqrt{2}z,r_2\times\sqrt{2}z} \begin{pmatrix} \frac{-\sqrt{2}(z^2+z)}{2} & \frac{3\sqrt{2}(z-z^2)}{2} & \sqrt{2}z & 0 \\ \frac{\sqrt{2}(z^2-z)}{5} & \frac{7\sqrt{2}(z+z^2)}{10} & 0 & \sqrt{2}z \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{-(z^{-1}+z^{-2})}{4\sqrt{2}} & \frac{(z^{-2}-z^{-1})}{2\sqrt{2}} \\ \frac{(z^{-1}-z^{-2})}{8\sqrt{2}} & \frac{(z^{-1}+z^{-2})}{4\sqrt{2}} \\ \frac{(30z^{-1}+5+5z^{-2})}{80\sqrt{2}} & \frac{(1-z^{-2})}{8\sqrt{2}} \\ \frac{3(z^{-2}-1)}{80\sqrt{2}} & \frac{(36z^{-1}-6-6z^{-2})}{80\sqrt{2}} \end{pmatrix}$$

$$= (G'_e(z)\ G'_o(z))_{2\times 4} \begin{pmatrix} \tilde{G}'_e(z)^* \\ \tilde{G}'_o(z)^* \end{pmatrix}_{4\times 2}.$$

Thus, from the definition of polyphase matrix, matrix coefficients of the corresponding multiwavelets can be obtained from the above $(G'_e(z)\ G'_o(z))$ and $(\tilde{G}'_e(z)\ \tilde{G}'_o(z))^*$, as follows:

$$G_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} -1 & 3 \\ -\frac{2}{5} & \frac{7}{5} \end{pmatrix}, \quad G_3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad G_4 = \begin{pmatrix} -1 & -3 \\ \frac{2}{5} & \frac{7}{5} \end{pmatrix},$$

$$\tilde{G}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{G}_1 = \begin{pmatrix} \frac{1}{16} & -\frac{3}{80} \\ \frac{1}{8} & -\frac{3}{40} \end{pmatrix}, \quad \tilde{G}_2 = \begin{pmatrix} -\frac{1}{4} & \frac{1}{8} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}, \quad \tilde{G}_3 = \begin{pmatrix} \frac{3}{8} & 0 \\ 0 & \frac{9}{20} \end{pmatrix},$$

$$\tilde{G}_4 = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{8} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}, \quad \tilde{G}_5 = \begin{pmatrix} \frac{1}{16} & \frac{3}{80} \\ -\frac{1}{8} & -\frac{3}{40} \end{pmatrix}.$$

The above matrix coefficients is consistent with the results as documented in Strela & Walden (1998). The graphs of $\phi_1(x)$, $\psi_1(x)$, $\phi_2(x)$, $\psi_2(x)$, $\tilde{\phi}_1(x)$, $\tilde{\psi}_1(x)$, $\tilde{\phi}_2(x)$ and $\tilde{\psi}_2(x)$ are shown in Figs 1 and 2, respectively.

EXAMPLE 2 Consider the biorthogonal sets of scaling vector functions $(\phi_1(x), \phi_2(x))$ and $(\tilde{\phi}_1(x), \tilde{\phi}_2(x))$ and their corresponding multiwavelets $(\psi_1(x), \psi_2(x))$ and $(\tilde{\psi}_1(x), \tilde{\psi}_2(x))$ presented in Goh & Yap (1998, pp. 153–156), where $\phi_1(x)$, $\tilde{\phi}_1(x)$, $\psi_2(x)$ and $\tilde{\psi}_2(x)$ are symmetric, while $\phi_2(x)$, $\tilde{\phi}_2(x)$, $\psi_1(x)$ and $\tilde{\psi}_1(x)$ are antisymmetric. All the scaling and multiwavelet functions have a support in $[-1, 1]$. The scaling coefficients are given in Goh & Yap (1998) as follows:

$$H(-1) = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ -1 & -\frac{2}{5} \end{pmatrix}, \quad H(0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad H(1) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{5} \\ 1 & -\frac{2}{5} \end{pmatrix},$$

$$\tilde{H}(-1) = \begin{pmatrix} \frac{1}{2} & \frac{5}{4} \\ -\frac{7}{16} & -\frac{35}{32} \end{pmatrix}, \quad \tilde{H}(0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \tilde{H}(1) = \begin{pmatrix} \frac{1}{2} & -\frac{5}{4} \\ \frac{7}{16} & -\frac{35}{32} \end{pmatrix}.$$

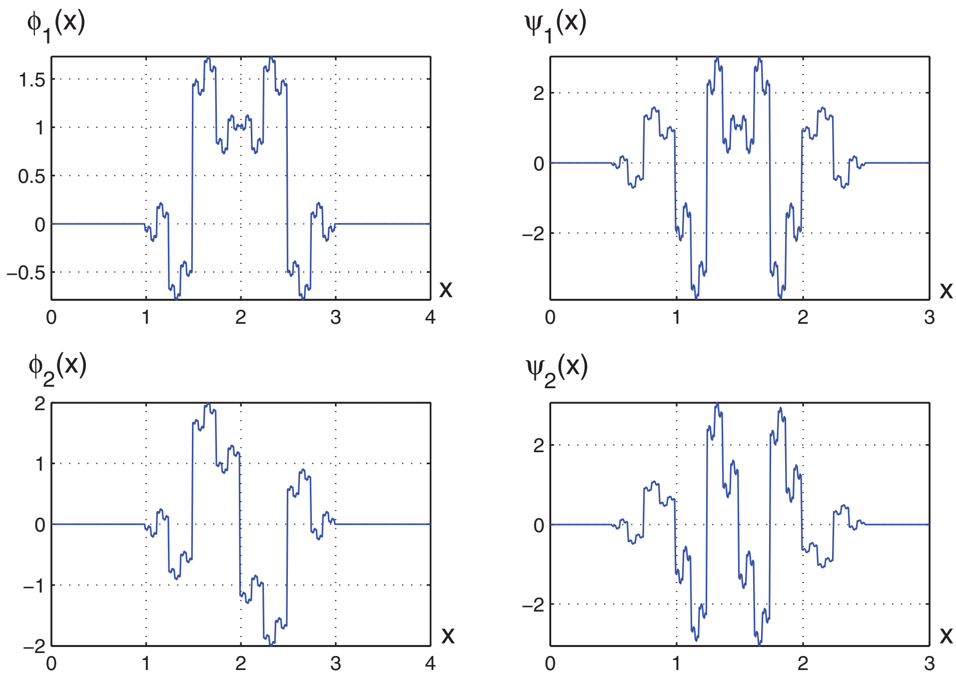

FIG. 1. Graphs of scaling functions and their corresponding multiwavelet functions of Example 1.

Solution. From the definition of the polyphase matrix described in Section 1, we have

$$H_e(z) = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{4} \end{pmatrix}, \quad H_o(z) = \begin{pmatrix} \frac{\sqrt{2}(z^{-1}+1)}{4} & \frac{\sqrt{2}(z^{-1}-1)}{10} \\ \frac{\sqrt{2}(1-z^{-1})}{2} & \frac{-\sqrt{2}(z^{-1}+1)}{5} \end{pmatrix},$$

$$\tilde{H}_e(z)^* = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{4} \end{pmatrix}, \quad \tilde{H}_o(z)^* = \begin{pmatrix} \frac{\sqrt{2}(z+1)}{4} & \frac{7\sqrt{2}(1-z)}{32} \\ \frac{5\sqrt{2}(z-1)}{8} & \frac{-35\sqrt{2}(z+1)}{64} \end{pmatrix}.$$

Step 1. Obviously, from Theorem 2.1, $\tilde{H}_e(z)^*$ is a unimodular square matrix since $\det \tilde{H}_e(z)^* = \frac{1}{4}$.

Step 2. Compute $G_e(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ by using the formulas in the second row of Table 1, as follows:

$$G_e(z) = -\tilde{H}_o(z)^*[\tilde{H}_e(z)^*]^{-1} = -4 \begin{pmatrix} \frac{\sqrt{2}(z+1)}{4} & \frac{7\sqrt{2}(1-z)}{32} \\ \frac{5\sqrt{2}(z-1)}{8} & \frac{-35\sqrt{2}(z+1)}{64} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}}{4} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

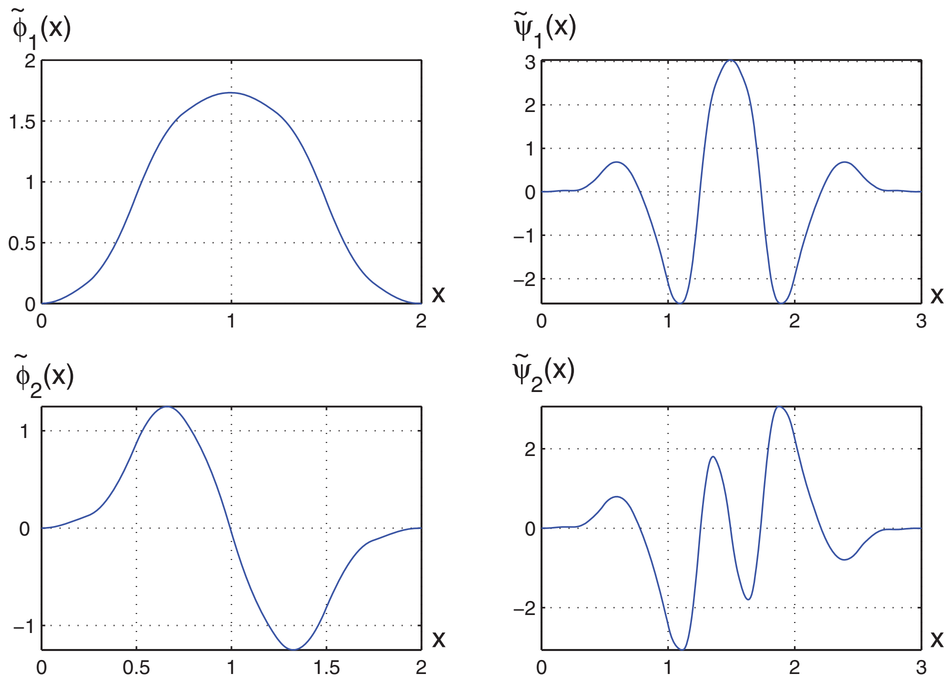

FIG. 2. Graphs of the dual scaling functions and their corresponding dual multiwavelet functions of Example 1.

$$= \begin{pmatrix} \frac{-(z+1)}{2} & \frac{7(z-1)}{8} \\ \frac{5(1-z)}{4} & \frac{35(z+1)}{16} \end{pmatrix},$$

$$\tilde{G}_e(z)^* = -\tilde{H}_e(z)^* H_o(z) = -\begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{4} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{2}(z^{-1}+1)}{4} & \frac{\sqrt{2}(z^{-1}-1)}{10} \\ \frac{\sqrt{2}(1-z^{-1})}{2} & -\frac{\sqrt{2}(z^{-1}+1)}{5} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{(z^{-1}+1)}{4} & \frac{(1-z^{-1})}{10} \\ \frac{(z^{-1}-1)}{4} & \frac{(z^{-1}+1)}{10} \end{pmatrix},$$

$$\tilde{G}_o(z)^* = I_{2\times 2} - \tilde{H}_o(z)^* H_o(z) = I_{2\times 2} - \begin{pmatrix} \frac{\sqrt{2}(z+1)}{4} & -\frac{7\sqrt{2}(z-1)}{32} \\ \frac{5\sqrt{2}(z-1)}{8} & -\frac{35\sqrt{2}(z+1)}{64} \end{pmatrix}$$

$$\cdot \begin{pmatrix} \dfrac{\sqrt{2}(z^{-1}+1)}{4} & \dfrac{\sqrt{2}(z^{-1}-1)}{10} \\ -\dfrac{\sqrt{2}(z^{-1}-1)}{2} & -\dfrac{\sqrt{2}(z^{-1}+1)}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{1}{32}(10+3z^{-1}+3z) & -\dfrac{3}{80}(z-z^{-1}) \\ -\dfrac{15}{64}(z^{-1}-z) & \dfrac{1}{32}(10-3z^{-1}-3z) \end{pmatrix}.$$

It can be verified that

$$Q(z)\tilde{Q}(z)^* = \begin{pmatrix} \dfrac{\sqrt{2}}{2} & 0 & \dfrac{\sqrt{2}(z^{-1}+1)}{4} & \dfrac{\sqrt{2}(z^{-1}-1)}{10} \\ 0 & \dfrac{\sqrt{2}}{4} & \dfrac{\sqrt{2}(1-z^{-1})}{2} & \dfrac{-\sqrt{2}(z^{-1}+1)}{5} \\ \dfrac{(z+1)}{-2} & \dfrac{7(z-1)}{8} & 1 & 0 \\ \dfrac{5(1-z)}{4} & \dfrac{35(z+1)}{16} & 0 & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} \dfrac{\sqrt{2}}{2} & 0 & \dfrac{(z^{-1}+1)}{-4} & \dfrac{(1-z^{-1})}{10} \\ 0 & \dfrac{\sqrt{2}}{4} & \dfrac{(z^{-1}-1)}{4} & \dfrac{(z^{-1}+1)}{10} \\ \dfrac{\sqrt{2}(z+1)}{4} & \dfrac{7\sqrt{2}(1-z)}{32} & \dfrac{(10+3z^{-1}+3z)}{32} & \dfrac{3(z^{-1}-z)}{80} \\ \dfrac{5\sqrt{2}(z-1)}{8} & \dfrac{35\sqrt{2}(z+1)}{-64} & \dfrac{15(z-z^{-1})}{64} & \dfrac{(10-3z^{-1}-3z)}{32} \end{pmatrix}$$

$$= I_{4\times 4}.$$

Step 3. Perform row–column product-preserving transformations for $(G_e(z)\ \ I_{2\times 2})_{2\times 4}$ and $(\tilde{G}_e(z)\ \ \tilde{G}_o(z))^*_{4\times 2}$ in order to obtain the same extension forms as reported in Goh & Yap (1998). Here, we only simply present the product-preserving transformation steps:

σ_1: (I) : $r_1 \times (-\frac{5}{2}) + r_2$; (II) : $c_2 \times \frac{5}{2} + c_1$;

σ_2: (I) : $r_2 \times \frac{1}{5}(1+z) + r_1$; (II) : $c_1 \times (\frac{-1}{5})(1+z) + c_2$;

σ_3: (I) : $r_1 \times z^{-1}$; (II) : $c_1 \times z$;

σ_4: (I) : $r_1 \times (\frac{-5}{2}) + r_2$; (II) : $c_2 \times \frac{5}{2} + c_1$;

σ_5: (I) : $r_1 \times \frac{2\sqrt{2}}{7}, r_2 \times (-\sqrt{2})$; (II) : $c_1 \times \frac{7}{2\sqrt{2}}, c_2 \times (\frac{-1}{\sqrt{2}})$.

After the above steps of product-preserving transformations, the resultant matrices $(G_e(z)\ G_o(z))_{2\times4}$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))^*_{4\times2}$ are

$$(G'_e(z)\ G'_o(z))_{2\times4}\begin{pmatrix}\tilde{G}'_e(z)^*\\ \tilde{G}'_o(z)^*\end{pmatrix}_{4\times2} = \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{-\sqrt{2}(1-z^{-1})}{7} & \frac{2\sqrt{2}(1+z^{-1})}{35} \\ \frac{-5}{\sqrt{2}} & 0 & \frac{5\sqrt{2}(1+z^{-1})}{4} & \frac{(z^{-1}-1)}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 0 & -\frac{\sqrt{2}}{10} \\ \frac{7\sqrt{2}}{8} & 0 \\ \frac{7\sqrt{2}(z-1)}{64} & \frac{\sqrt{2}(1+z)}{20} \\ \frac{35\sqrt{2}(1+z)}{128} & \frac{\sqrt{2}(z-1)}{8} \end{pmatrix}.$$

The above two matrices align with $(G_e(z)\ G_o(z))_{2\times4}$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))^*_{4\times2}$ as addressed in Goh & Yap (1998). The graphs of $\phi_1(x)$, $\psi_1(x)$, $\phi_2(x)$, $\psi_2(x)$, $\tilde{\phi}_1(x)$, $\tilde{\psi}_1(x)$, $\tilde{\phi}_2(x)$ and $\tilde{\psi}_2(x)$ are shown in Figs 3 and 4, respectively.

Now, another set of formulas (the first row in Table 2) is employed to compute this example for the purpose of comparing the computational cost. It shows that appropriately selecting formulas from Tables 1 and 2 can save computational cost greatly. Here, we can note that after $G_e(z)$, $G_o(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ are obtained by using the first row formulas in Table 2, only two steps of product-preserving transformations are needed to obtain the desired forms.

Step 1. Obviously, we can verify that $\tilde{H}_e(z)^*$ and $\tilde{H}_o(z)^*$ are unimodular square matrices since $\det\tilde{H}_e(z)^* = \frac{1}{4}$ and $\det\tilde{H}_o(z)^* = -\frac{35}{32}z$.

Step 2. Compute $G_e(z)$, $G_o(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ by using the formulas in the first row of Table 2, as follows:

$$G_e(z) = -[\tilde{H}_e(z)^*]^{-1} = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & -2\sqrt{2} \end{pmatrix},$$

$$G_o(z) = [\tilde{H}_o(z)^*]^{-1} = \begin{pmatrix} \frac{\sqrt{2}(1+z^{-1})}{2} & \frac{\sqrt{2}(z^{-1}-1)}{5} \\ \frac{4\sqrt{2}(1-z^{-1})}{7} & \frac{-8\sqrt{2}(1+z^{-1})}{35} \end{pmatrix},$$

$$\tilde{G}_e(z)^* = -\tilde{H}_e(z)^* H_o(z)\tilde{H}_o(z)^* = \begin{pmatrix} \frac{-\sqrt{2}}{4} & 0 \\ 0 & \frac{-7\sqrt{2}}{32} \end{pmatrix},$$

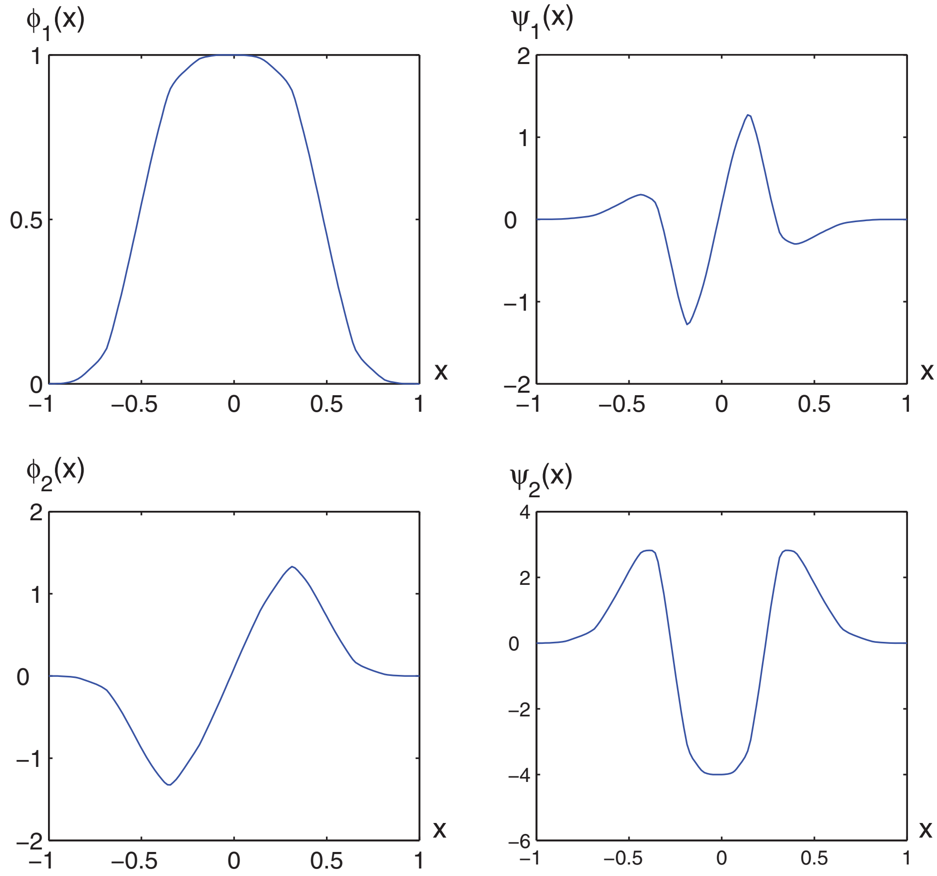

FIG. 3. Graphs of scaling functions and their corresponding multiwavelet functions of Example 2.

$$\tilde{G}_o(z)^* = \tilde{H}_o(z)^* H_e(z) \tilde{H}_e(z)^* = \begin{pmatrix} \dfrac{\sqrt{2}(1+z)}{8} & \dfrac{7\sqrt{2}(1-z)}{256} \\ \dfrac{5\sqrt{2}(z-1)}{16} & \dfrac{-35\sqrt{2}(1+z)}{512} \end{pmatrix}.$$

Step 3. In order to obtain the same extension result as that in Goh & Yap (1998), we only need to perform the following simple row–column product-preserving transformations for $(G_e(z)\ G_o(z))_{2\times 4}$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))^*_{4\times 2}$:

σ_1: (I) : $r_1 \leftrightarrow r_2$; (II) : $c_1 \leftrightarrow c_2$;

σ_2: (I) : $r_1 \times (-\frac{1}{4}), r_2 \times \frac{5}{2}$; (II) : $c_1 \times (-4), c_2 \times \frac{2}{5}$.

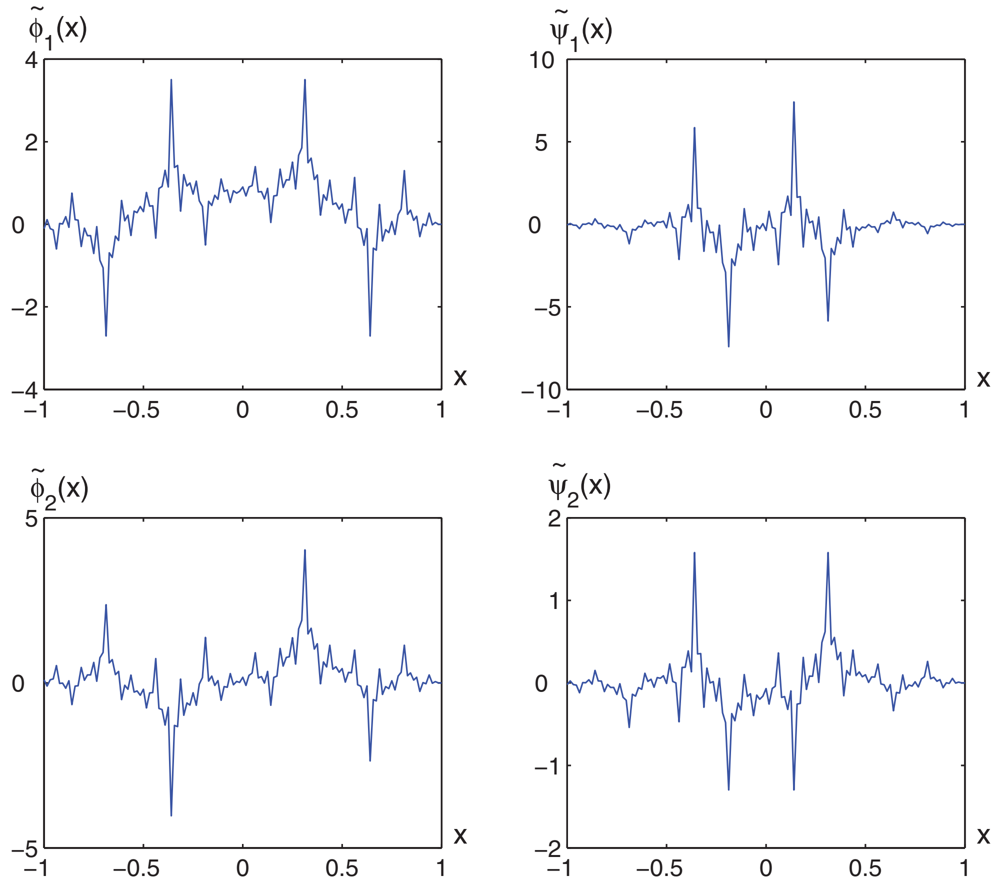

FIG. 4. Graphs of the dual scaling functions and their corresponding dual multiwavelet functions of Example 2.

EXAMPLE 3 The biorthogonal multiwavelet in this example is constructed in Hardin & Marasovich (1999, pp. 48–51) by using fractal interpolation functions. The supports of the scaling functions and multiwavelet functions are in $[-1,1]$. The scaling functions are symmetric, and the associated multiwavelet functions are symmetric/antisymmetric. The scaling coefficients are given in Hardin & Marasovich (1999) as follows:

$$H_{-2}=\begin{pmatrix}0 & -\frac{1}{6\sqrt{3}}\\ 0 & 0\end{pmatrix},\quad H_{-1}=\begin{pmatrix}-\frac{1}{6} & \frac{5}{6\sqrt{3}}\\ 0 & 0\end{pmatrix},\quad H_{0}=\begin{pmatrix}1 & \frac{5}{6\sqrt{3}}\\ 0 & \frac{2}{3}\end{pmatrix},\quad H_{1}=\begin{pmatrix}-\frac{1}{6} & -\frac{1}{6\sqrt{3}}\\ \frac{2}{\sqrt{3}} & \frac{2}{3}\end{pmatrix},$$

$$\tilde{H}_{-2}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\quad \tilde{H}_{-1}=\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ 0 & 0\end{pmatrix},\quad \tilde{H}_{0}=\begin{pmatrix}1 & \frac{\sqrt{3}}{2}\\ 0 & \frac{1}{2}\end{pmatrix},\quad \tilde{H}_{1}=\begin{pmatrix}-\frac{1}{2} & 0\\ \frac{2}{\sqrt{3}} & \frac{1}{2}\end{pmatrix}.$$

Solution. From the definition of the polyphase matrix described in Section 1, we have

$$H_e(z) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{(5-z^{-1})}{6\sqrt{6}} \\ 0 & \frac{\sqrt{2}}{3} \end{pmatrix}, \quad H_o(z) = \begin{pmatrix} -\frac{(z^{-1}+1)}{6\sqrt{2}} & \frac{(5z^{-1}-1)}{6\sqrt{6}} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{2}}{3} \end{pmatrix},$$

$$\tilde{H}_e(z)^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{pmatrix}, \quad \tilde{H}_o(z)^* = \begin{pmatrix} -\frac{1}{2\sqrt{2}}(1+z) & \frac{2}{\sqrt{6}} \\ \frac{\sqrt{3}}{2\sqrt{2}}z & \frac{1}{2\sqrt{2}} \end{pmatrix}.$$

Step 1. Since $\det H_o(z) = -\frac{1}{3}z^{-1}$, $\det \tilde{H}_e(z)^* = \frac{1}{4}$, $H_o(z)$ and $\tilde{H}_e(z)^*$ are unimodular square matrices over $R[z]$, it follows that $G_e(z)$, $G_o(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ can be calculated directly.

Step 2. Compute $G_e(z)$, $G_o(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ by using the formulas listed in the fifth row of Table 2, as follows:

$$G_e(z) = [\tilde{H}_e(z)^*]^{-1} - H_e(z) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{(z^{-1}-5)}{6\sqrt{6}} \\ -\sqrt{6} & \frac{5\sqrt{2}}{3} \end{pmatrix},$$

$$G_o(z) = -H_o(z) = \begin{pmatrix} \frac{(z^{-1}+1)}{6\sqrt{2}} & \frac{(1-5z^{-1})}{6\sqrt{6}} \\ \frac{-\sqrt{6}}{3} & \frac{-\sqrt{2}}{3} \end{pmatrix},$$

$$\tilde{G}_e(z)^* = \tilde{H}_e(z)^* = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{pmatrix},$$

$$\tilde{G}_o(z)^* = -[H_o(z)]^{-1}H_e(z)\tilde{H}_e(z)^* = \begin{pmatrix} \frac{(3z-1)}{2\sqrt{2}} & \frac{(z-1)}{2\sqrt{6}} \\ -\frac{3\sqrt{3}}{2\sqrt{2}}z & -\frac{1}{2\sqrt{2}}z \end{pmatrix}.$$

Step 3. Perform row–column product-preserving transformations for $(G_e(z) \ \ G_o(z))_{2\times 4}$ and $(\tilde{G}_e(z) \ \ \tilde{G}_o(z))^*_{4\times 2}$ to obtain the same multiwavelet matrix coefficients as that in Hardin & Marasovich (1999).

σ_1: (I): $r_1 \times \sqrt{12} + r_2$; (II): $c_2 \times (-\sqrt{12}) + c_1$;

σ_2: (I): $r_1 \times -1$, $r_2 \times (-\frac{1}{\sqrt{6}})$; (II): $c_1 \times (-1)$, $c_2 \times (-\sqrt{6})$.

After the above two simple steps of product-preserving transformations, the resultant matrices $(G_e(z)\ G_o(z))_{2\times 4}$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))^*_{4\times 2}$ are

$$(G_e(z)\ G_o(z))_{2\times 4}\cdot\begin{pmatrix}\tilde{G}_e(z)^*\\ \tilde{G}_o(z)^*\end{pmatrix}_{4\times 2}=\begin{pmatrix}\frac{-1}{\sqrt{2}} & \frac{(5-z^{-1})}{6\sqrt{6}} & \frac{-(1+z^{-1})}{6\sqrt{2}} & \frac{(5z^{-1}-1)}{6\sqrt{6}}\\ 0 & \frac{-(5+z^{-1})}{6\sqrt{3}} & \frac{(1-z^{-1})}{6} & \frac{(1+5z^{-1})}{6\sqrt{3}}\end{pmatrix}$$

$$\cdot\begin{pmatrix}\frac{-1}{\sqrt{2}} & 0\\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{-\sqrt{3}}{2}\\ \frac{-(z+1)}{2\sqrt{2}} & \frac{(1-z)}{2}\\ \frac{\sqrt{3}}{2\sqrt{2}}z & \frac{\sqrt{3}}{2}z\end{pmatrix},$$

which is consistent with the results as documented in Hardin & Marasovich (1999). The graphs of $\phi_1(x)$, $\psi_1(x)$, $\phi_2(x)$, $\psi_2(x)$, $\tilde{\phi}_1(x)$, $\tilde{\psi}_1(x)$, $\tilde{\phi}_2(x)$ and $\tilde{\psi}_2(x)$ are shown in Figs 5 and 6, respectively.

EXAMPLE 4 Here, the biorthogonal multiwavelet (*2/4 SABMF*) is reconstructed as presented in Tan *et al.* (1999). Both the scaling and wavelet functions are symmetric/antisymmetric about $\frac{1}{2}$. The scaling coefficients were given in Tan *et al.* (1999) as follows:

$$H_0=\begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix},\quad H_1=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix},\quad \tilde{H}_{-1}=\begin{pmatrix}0 & \frac{1}{8}\\ 0 & -\frac{1}{8}\end{pmatrix},$$
$$\tilde{H}_0=\begin{pmatrix}1 & \frac{1}{8}\\ -1 & \frac{1}{8}\end{pmatrix},\quad \tilde{H}_1=\begin{pmatrix}1 & -\frac{1}{8}\\ 1 & \frac{1}{8}\end{pmatrix},\quad \tilde{H}_2=\begin{pmatrix}0 & -\frac{1}{8}\\ 0 & -\frac{1}{8}\end{pmatrix}.$$

Solution. **Step 1.** The polyphase matrices of the scaling coefficients are

$$H_e(z)=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0\\ \frac{-1}{\sqrt{2}} & 0\end{pmatrix},\quad H_o(z)=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & 0\end{pmatrix},$$
$$\tilde{H}_e(z)=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1-z}{8\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1-z}{8\sqrt{2}}\end{pmatrix},\quad \tilde{H}_o(z)=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{z^{-1}-1}{8\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1-z^{-1}}{8\sqrt{2}}\end{pmatrix}.$$

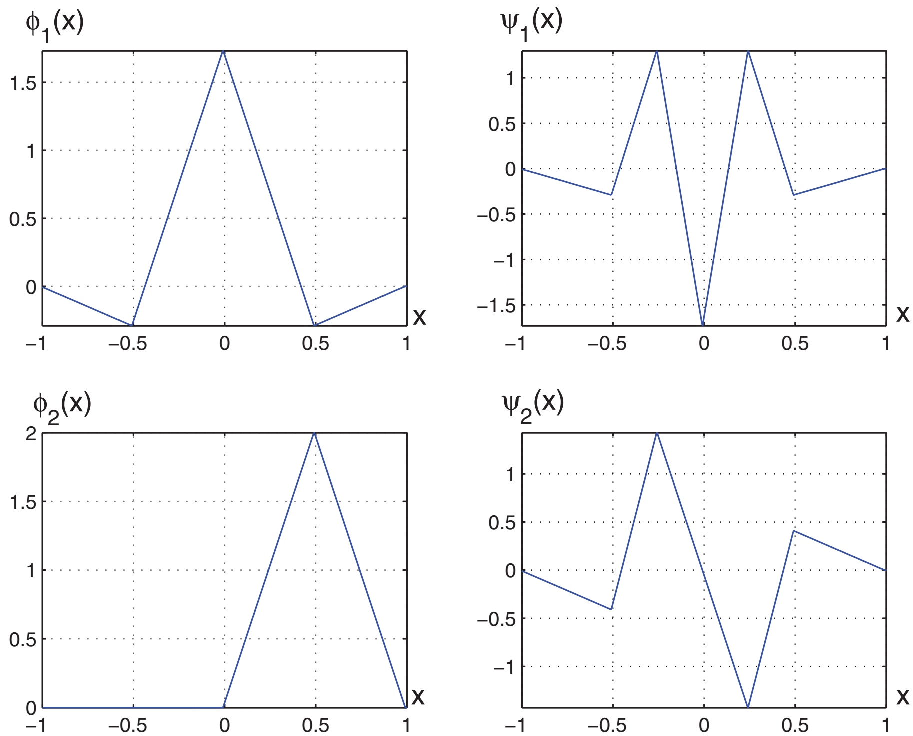

FIG. 5. Graphs of scaling functions and their corresponding multiwavelet functions of Example 3.

There is no unimodular matrix among these four matrices. Thus, column–row product-preserving transformations need to be performed on $H(z)$ and $\tilde{H}(z)^*$. Here, only one step of column–row product-preserving transformation needs to be performed:

$$\sigma : (\mathrm{I}) : c_2 \leftrightarrow c_3; \quad (\mathrm{II}) : r_2 \leftrightarrow r_3,$$

i.e.

$$H(z)\tilde{H}(z)^* = (H_e(z)\ H_o(z)) \begin{pmatrix} \tilde{H}_e(z)^* \\ \tilde{H}_o(z)^* \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1-z^{-1}}{8\sqrt{2}} & \frac{1-z^{-1}}{8\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{z-1}{8\sqrt{2}} & \frac{1-z}{8\sqrt{2}} \end{pmatrix},$$

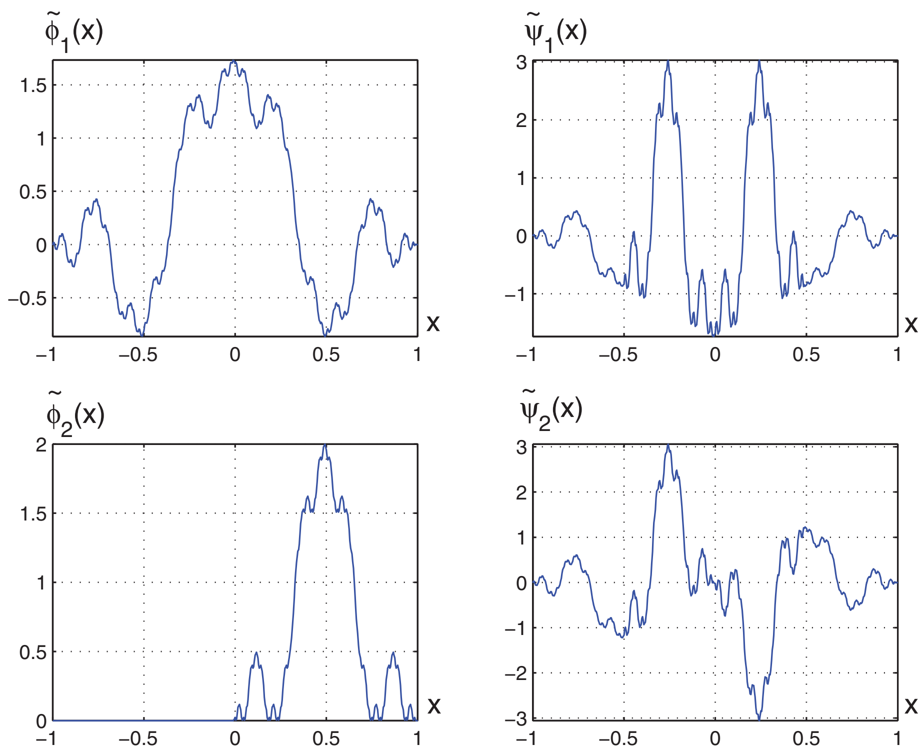

FIG. 6. Graphs of the dual scaling functions and their corresponding dual multiwavelet functions of Example 3.

$$\xrightarrow[\text{(II)}:r_2\leftrightarrow r_3]{\text{(I)}:c_2\leftrightarrow c_3} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1-z^{-1}}{8\sqrt{2}} & \frac{1-z^{-1}}{8\sqrt{2}} \\ \frac{z-1}{8\sqrt{2}} & \frac{1-z}{8\sqrt{2}} \end{pmatrix}$$

$$= (H_e'(z)\ H_o'(z)) \begin{pmatrix} \tilde{H}_e'(z)^* \\ \tilde{H}_o'(z)^* \end{pmatrix}.$$

We see that $H_e'(z)$ and $\tilde{H}_e'(z)^*$ are unimodular matrices since $\det H_e'(z) = \det \tilde{H}_e'(z)^* = 1$.

Step 2. Compute $G_e'(z)$, $G_o'(z)$, $\tilde{G}_e'(z)^*$ and $\tilde{G}_o'(z)^*$ by using the formulas in the second row of Table 1 as follows:

$$G_e'(z) = -\tilde{H}_o'(z)^*[\tilde{H}_e'(z)^*]^{-1} = \begin{pmatrix} 0 & \frac{(z^{-1}-1)}{8} \\ \frac{(1-z)}{8} & 0 \end{pmatrix}, \quad G_o'(z) = I_2,$$

$$\tilde{G}'_e(z)^* = -\tilde{H}'_e(z)^* H'_o(z) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{G}'_o(z)^* = I_2 - \tilde{H}'_o(z)^* H'_o(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Step 3. Transform $(G'_e(z)\ G'_o(z))_{p\times 2p}$ and $(\tilde{G}'_e(z)\ \tilde{G}'_o(z))^*_{2p\times p}$ by the column–row product-preserving transformation σ^{-1} : (I) : $c_2 \leftrightarrow c_3$; (II) : $r_2 \leftrightarrow r_3$. Then we have

$$(G_e(z) \quad G_o(z)) \begin{pmatrix} \tilde{G}_e(z)^* \\ \tilde{G}_o(z)^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & \dfrac{(z^{-1}-1)}{8} & 0 \\ \dfrac{(1-z)}{8} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Step 4. Perform row–column product-preserving transformations for $(G_e(z)\ \ G_o(z))_{2\times 4}$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))^*_{4\times 2}$ to obtain the same multiwavelet matrix coefficients as that in Tan *et al.* (1999). Here, we only present the product-preserving transformation steps.

σ_1 : (I) : $r_2 \times (-1) + r_1$; (II) : $c_1 + c_2$;

σ_2 : (I) : $r_1 \times (-1)$; $r_2 \times 2$; (II) : $c_1 \times (-1)$; $c_2 \times \frac{1}{2}$;

σ_3 : (I) : $r_1 \times (-1) + r_2$; (II) : $c_2 + c_1$.

After the above three steps of product-preserving transformations, the resultant matrices $(G_e(z)\ G_o(z))_{2\times 4}$ and $(\tilde{G}_e(z)\ \tilde{G}_o(z))^*_{4\times 2}$ are

$$(G_e(z)\ G_o(z)) \begin{pmatrix} \tilde{G}_e(z)^* \\ \tilde{G}_o(z)^* \end{pmatrix} = \begin{pmatrix} \dfrac{1-z}{8\sqrt{2}} & -\dfrac{1}{\sqrt{2}} & \dfrac{1-z^{-1}}{8\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ \dfrac{1-z}{8\sqrt{2}} & \dfrac{1}{\sqrt{2}} & \dfrac{z^{-1}-1}{8\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \\ 0 & 0 \\ \dfrac{1}{\sqrt{2}} & \dfrac{1}{\sqrt{2}} \end{pmatrix},$$

which align with the results as documented in Tan *et al.* (1999).

6. Applications to denoising of 1D signals

In this section, we compare the numerical performance of the multiwavelets showed in the examples with the Daubechies D_4 scalar wavelet. D_4 wavelet is a commonly used wavelet in signal processing. These comparisons are performed for five test signals (Blocks, Bumps, Heavy Sine, Doppler and Quad-chirp) given by MATLAB. All these five signals and noisy signals can be obtained by the function '*wnoise*' in MATLAB.

Suppose that a signal of interest f has been corrupted by noise; then a signal g is observed as

$$g[n] = f[n] + \sigma z[n], \quad n = 1, \ldots, N,$$

where $z[n]$ is unit-variance Gaussian white noise. So, the key problem is how to recover f from the samples $g[n]$ as best as possible. Donoho & Johnstone (1994) (see also Donoho (1995)) proposed a solution via wavelet shrinkage or soft thresholding in the wavelet domain. Donoho and Johnstone's algorithm offers the advantages of smoothness and adaptation. Wavelet shrinkage is *smooth* in the sense that the denoised estimate $\hat{f}$ has a very high probability of being as smooth as the original signal f, in a variety of smoothness spaces (Sobolev, Holder, etc.). Heuristically, wavelet shrinkage has the advantage of not adding 'bumps' or false oscillations in the process of removing noise due to the local and smoothness-preserving nature of the wavelet transform.

As in the scalar case, low-pass filter H and high-pass filter G consist of coefficients, corresponding to the scaling and wavelet functions, respectively. But now these coefficients are $p \times p$ matrices (in this paper, $p = 2$) and during the convolution step they must multiply vectors (instead of numbers). This means that multifilter banks need p input rows. Usually, there is only one input signal $f[n]$ at the beginning, so some kind of preprocessing of the data must be done before the implementation of a multifilter bank. Also, in the reconstruction, multiwavelets require a postprocessing. The postprocessing is just an inversion of the preprocessing. In our case $p = 2$ and two data streams enter the multifilter. To create them from an ordinary single-stream input of length N, there are several possibilities:

(1) Separate odd and even samples (in 1D), or use adjacent rows of the image (in 2D).

(2) Repeat the input stream to produce two length N streams.

(3) Create a consistent approximation that yields two length $N/2$ streams, and a 'de-approximation' that returns a length N stream.

Readers who are interested in a more detailed explanation of the preprocessing may refer to Strela & Walden (1998) and Tham *et al.* (2000). Here, we simply use the 'repeated row' for denoising. Although this procedure introduces oversampling of the data by a factor of 2, this scheme is convenient to implement and suitable for all multifilters. On the other hand, it is known that oversampled data representations are useful for feature extraction.

The decomposition and reconstruction of multiwavelets are similar with the scalar wavelet. Figure 7 depicts a 1-level subband decomposition and reconstruction framework for a discrete biorthogonal multiwavlet transform. The left half of the figure represents a 1-level multiwavelet decomposition. First, an ordinary single-stream signal f is sent to the preprocessing and two data streams are obtained. Then the two data streams are decomposed by the matrix low-pass filters H and G, respectively, to generate the next lower resolution. This is followed by subsampling by a factor of 2 to preserve compact representation of the input signal in a 2-band filtering. For octave-bandwidth decomposition, only the low-pass subbands can be decomposed iteratively to produce subsequent lower resolutions. Graphically, a J-level decomposition consists of a cascade of J such 1-level decompositions, each operating on the low-pass subbands of the previous resolution. The right half of Fig. 7 represents the corresponding 1-level multiwavelet reconstruction. The subbands are first upsampled by a factor of 2 before they are filtered by the synthesis matrix filters to recover the original single-stream signal. Finally, the reconstructed two data streams pass the postprocessing and the recovered signal $\hat{f}$ is obtained. Both the matrix filter pairs $\{H, G\}$ and $\{\tilde{H}, \tilde{G}\}$ can be used as the analysis or synthesis filters. But in general, the smoother one is used as the synthesis filter so that the reconstructed signal can be more smooth.

The denoising algorithm is described as follows:

(1) Apply the cascade algorithm to get the wavelet coefficients corresponding to $g[n]$.

(2) Choose threshold $\tau = \sqrt{2\log(N)}\sigma$ and apply soft thresholding to the wavelet coefficients, where N is the sampling points of the $g[n]$ and σ is the Standard Deviation of the white noise. The scaling coefficients entirely remained alone as these carry low-frequency *smooth* information.

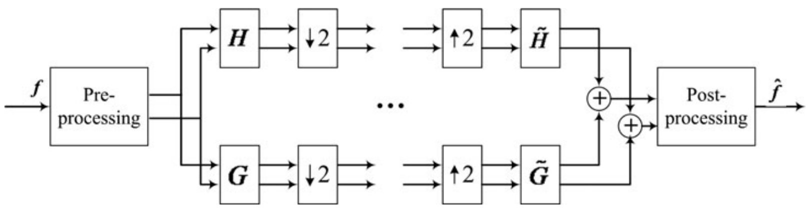

FIG. 7. One level of decomposition and reconstruction by using the biorthogonal multiwavelet filter bank.

TABLE 3 *RMSE comparison of the multiwavelets and D4 scalar wavelet for different test signals*

Signals	BiGHM	SaySong	Hardin	2/4SABMF	D4
Blocks	0.8142	1.1904	1.2540	1.3575	1.2796
Bumps	0.6717	1.1603	0.9648	1.2198	1.2469
Heavy Sine	0.3992	0.5732	0.6592	0.399	0.6123
Doppler	0.5982	1.0293	1.0358	0.9533	1.1422
Quadchirp	1.5513	1.6219	1.8891	1.9965	2.1856

(3) Invert the cascade algorithm to get denoised signal $\hat{f}[n]$.

Here, $N = 512$, the signal-to-noise ratio of the noisy signals is set to 5 dB and the decomposition level is 5. The root mean squared error (RMSE) was computed for each signal processed by each multiwavelet. The RMSE is defined as $\mathrm{RMSE} = \sqrt{\sum_{k=1}^{N}(f[n] - \hat{f}[n])^2/N}$. The results of a typical experiment are shown in Table 3. It can be noted that most of the time the multiwavelets shown in Section 5 performed better than the D_4 wavelet in the experiments with different test signals.

7. Conclusion

This paper proposed a novel abstract algebraic method for matrix extension in the construction of compactly supported biorthogonal multiwavelets. By investigating the properties of canonical-form polyphase matrices of the scaling vector functions, several solution sets are developed with explicit formulas of the polyphase matrix extension. Furthermore, the relationship between any two different extensions resulted from a same extension problem can be established through finite steps of product-preserving transformations. As a result, the complete solution set for a given matrix extension problem of compactly supported biorthogonal multiwavelets can be obtained. All these achievements are based on two theorems we proved: for any given $H(z)$ and $\tilde{H}(z)^*$, they can always be transformed into a canonical form; then the polyphase matrices $G_e(z)$, $G_o(z)$, $\tilde{G}_e(z)^*$ and $\tilde{G}_o(z)^*$ can be solved from $Q(z)\tilde{Q}(z)^* = I_{2p\times 2p}$ and be matrices with Laurent polynomial entries for sure. Construction examples showed that the proposed abstract algebraic approach is straightforward and explicit. Finally, an application of biorthogonal multiwavelet to denoise 1D signals was given. The experimental results showed that the multiwavelets generally outperform scalar wavelets under different test signals.

Acknowledgements

The authors would like to thank the anonymous reviewers for many valuable comments which contributed to the improvement of this paper.

Funding

This work was supported by the National Natural Science Foundation of China (61272028, 61104078, 61073079, 60802045); Fundamental Research Funds for the Central Universities of China (2011JBM223, 2013JBZ003, 2010l2200159, 2010l2200110); Beijing Natural Science Foundation (4113075); Specialized Research Fund for the Doctoral Program of Higher Education of China (20110162120045, 20100162120019); Research Fund for the Doctoral Program of Higher Education of China (20120009110008); Program for New Century Excellent Talents in University (NCET-12-0768); National Key Technology R&D Program of China (2012BAH01F03); National Basic Research (973) Program of China (2011CB302203); Foundation of Key Laboratory of System Control and Information Processing (SCIP2011009) and Science and Technology Program of Hunan Province (2011ck3066).

References

Chui, C. K. & Lian, J. (1996) A study on orthonormal multiwavelets. *Appl. Numer. Math.*, **20**, 273–298.

Daubechies, I. (1992) *Ten Lectures on Wavelets*. Philadelphia: Society for Industrial and Applied Mathematics.

Donoho, D. L. (1995) De-noising by soft-thresholding. *IEEE Trans. Inf. Theory*, **41**, 613–627.

Donoho, D. L. & Johnstone, I. M. (1994) Ideal spatial adaptation via wavelet shrinkage. *Biometrika*, **81**, 425–455.

Donovan, G., Geronimo, J. S., Hardin, D. P. & Massopust, P. R. (1996) Construction of orthogonal wavelets using fractal interpolation functions. *SIAM J. Math. Anal.*, **27**, 1158–1192.

Geronimo, J. S., Hardin, D. P. & Massopust, P. R. (1994) Fractal functions and wavelet expansions based on several scaling functions. *J. Approx. Theory*, **78**, 373–401.

Goh, S. S., Jiang, Q. T. & Xia, T. (2000) Construction of biorthogonal multiwavelets using the lifting scheme. *Appl. Comput. Harmon. Anal.*, **9**, 336–352.

Goh, S. S. & Yap, V. B. (1998) Matrix extension and biorthogonal multiwavelet construction. *Linear Algebra Appl.*, **269**, 139–157.

Goodman, T. N. T., Lee, S. L. & Tang, W. S. (1990) Wavelet in wandering subspace. *Trans. Amer. Math. Soc.*, **338**, 639–654.

Hardin, D. P. & Marasovich, S. A. (1999) Biorthogonal multiwavelets on $[-1, 1]$. *Appl. Comput. Harmon. Anal.*, **7**, 34–53.

Jacobson, N. (1974) *Basic Algebra I*. New York: W. H. Freeman and Company.

Lawton, W., Lee, S. L. & Shen, Z. (1996) An algorithm for matrix extension and wavelet construction. *Math. Comp.*, **65**, 723–737.

Roman, S. (1997) *Advanced Linear Algebra*. China, Beijing: World Publishing Corporation.

Strela, V. & Walden, A. T. (1998) Signal and image denoising via wavelet thresholding: orthogonal and biorthogonal, scalar and multiple wavelet transforms. *Statistics Section Technical Report TR-98-01*. Dept. of Mathematics, Imperial College of Science, Technology & Medicine, March.

Tan, H. H., Shen, L. X. & Tham, J. Y. (1999) New biorthogonal multiwavelets for image compression. *Signal Process.*, **79**, 45–65.

Tham, J. Y, Shen, L. X., Lee, S. L. & Tan, H. H. (2000) A general approach for analysis and application of discret multiwavelet transforms. *IEEE Trans. Signal Process.*, **48**, 457–464.

Waerden, B. L. V. (1978) *Algebra II*. China, Beijing: Science Press (in Chinese) (Translated by Cao, X. H., Zeng, K. C. & Hao, B. X.).

Appendix A. Proof of Lemma 2.3

$\forall\, a_{ij}(z) \neq 0$; because $a_{ij}(z)$ is a Laurent polynomial in z, there exists a non-zero term with the minimal degree of z.

Let N_k be the minimal degree of z for all non-zero entries in the kth row of $A(z)$. Now, for $1 \leqslant k \leqslant n$ and $\forall\, k$, multiplying the kth row of $A(z)$ by z^{-N_k} yields a matrix $B(z) = (b_{ij}(z))_{n\times n}$ over a Euclidean subring $P[z]$ of $R[z]$. Obviously, every entry in $B(z)$ is a polynomial in z and $B(z)$ also remains as a unimodular square matrix over $R[z]$ by Proposition 2.3.

Appendix B. Proof of Theorem 2.2

From Lemmas 2.2 and 2.3, $A(z)$ can be converted into the diagonal form

$$\mathrm{diag}(d_1(z), \ldots, d_n(z)) \quad \text{where } d_i(z) \neq 0 \text{ and } d_i(z)|d_j(z), \text{ if } i < j,$$

by finite steps of elementary transformations, i.e. there are elementary matrices $P_1, \ldots, P_r$ (corresponding to the row-elementary transformations) and $Q_1, \ldots, Q_s$ (corresponding to the column-elementary transformations) such that $P_r \cdots P_1 A(z) Q_1 \cdots Q_s = \mathrm{diag}(d_1(z), \ldots, d_n(z))$. Because $A(z) \in \mathrm{GL}_n(R[z])$, it is true that $\mathrm{diag}(d_1(z), \ldots, d_n(z)) \in \mathrm{GL}_n(R[z])$ by Proposition 2.3. Therefore, $d_i(z)$, $i = 1, \ldots, n$, must be an invertible element of $R[z]$, thus, a unit.

Perform the following row-elementary transformations of Type II for $\mathrm{diag}(d_1(z), \ldots, d_n(z))$:

For $1 \leqslant k \leqslant n$ and $\forall k$, multiplying the kth row by $d_k^{-1}(z)$, we have that $\mathrm{diag}(d_1(z), \ldots, d_n(z))$ can be converted into $I_{n\times n}$, i.e.

$$\begin{aligned} &D_n(d_n^{-1}(z)) \cdots D_1(d_1^{-1}(z)) P_r \cdots P_1 A(z) Q_1 \cdots Q_s = I_{n\times n} \\ &\quad \Rightarrow D_n(d_n^{-1}(z)) \cdots D_1(d_1^{-1}(z)) P_r \cdots P_1 A(z) = I_{n\times n} Q_s^{-1} \cdots Q_1^{-1} = Q_s^{-1} \cdots Q_1^{-1} \cdot I_{n\times n}. \end{aligned}$$

Therefore,

$$Q_1 \cdots Q_s D_n^{-1}(d_n(z)) \cdots D_1^{-1}(d_1(z)) P_r \cdots P_1 A(z) = I_{n\times n},$$

which is equivalent to the fact that $A(z)$ can be converted into $I_{n\times n}$ through finite steps of row-elementary transformations.

Appendix C. Proof of Theorem 2.3

From Proposition 2.3, $A_1(z)$ and $B_1(z)$ remain as two matrices over $R[z]$. From Proposition 2.2, performing a row–column product-preserving transformation for $A(z)$ and $B(z)$ amounts to left multiplication of $A(z)$ by an elementary matrix $P_{m\times m}$ and, simultaneously, right multiplication of $B(z)$ by $P_{m\times m}^{-1}$, i.e. $A_1(z) = P_{m\times m} A(z)$, $B_1(z) = B(z) P_{m\times m}^{-1}$. Thus,

$$A_1(z) B_1(z) = P_{m\times m} A(z) B(z) P_{m\times m}^{-1} = P_{m\times m} I_{m\times m} P_{m\times m}^{-1} = I_{m\times m}.$$

Similarly, performing a column–row product-preserving transformation for $A(z)$ and $B(z)$, this theorem also holds.